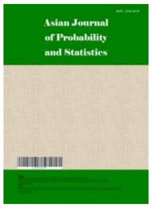

Asian Journal of Probability and Statistics

13(3): 1-12, 2021; Article no.AJPAS.69405
ISSN: 2582-0230

A Study on Various Transformation Method of Weibull Distribution: A Review

Seema Chettri[1] and Bhanita Das[1*]

[1]Department of Statistics, North Eastern Hill University, Shillong, India.

Authors' contributions

This work was carried out in collaboration among the authors. Author SC designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Author BD managed the literature searches. Both the authors read and approved the final manuscript.

Article Information

DOI: 10.9734/AJPAS/2021/v13i330307
Editor(s):
(1) Dr. Manuel Alberto M. Ferreira, Lisbon University, Portugal.
Reviewers:
(1) Collins Odhiambo, Strathmore University, Kenya.
(2) Ashok K. Singh, University of Nevada, USA.
Complete Peer review History: http://www.sdiarticle4.com/review-history/69405

Review Article

Received 25 March 2021
Accepted 01 June 2021
Published 08 June 2021

Abstract

In this article a brief summary of some recent developments of Weibull lifetime models has been presented for a quick overview. Various extensions of the Weibull models and the properties of the extended Weibull distribution have been discussed. A brief discussion about the characteristics and shape behaviour has been presented in the tabular form. Finally, some future research topics have been given.

Keywords: *Weibull distribution; exponentiated Weibull distribution; modified Weibull distribution; flexible Weibull distribution; transformed Weibull distribution; alpha power transformation; reliability function; hazard rate function and cumulative hazard rate function.*

NOTATION

F(t) : *Cumulative distribution function (cdf)*
f(t) : *Probability density function (pdf)*
R(t) : *Reliability function*
h(t) : *Hazard rate function*
H(t) : *Cumulative hazard rate function*

**Corresponding author: Email: bhanitadas83@gmail.com;*

ACRONYM

IFR	*: Increasing failure rate*
DFR	*: Decreasing failure rate*
UFR	*: Unimodal failure rate*
BT	*: Bathtub*
MBT	*: Modified bathtub*
UBT	*: Upside-down bathtub*
IDI	*: Increasing -Decreasing-Increasing*
DID	*: Decreasing-Increasing-Decreasing*
EG-FEW	*: Exponentiated Generalized- Flexible Weibull Extension*
WEW	*: Weibull Exponentiated Weibull*
NMW	*: New Modified Weibull*
RNMW	*: Reduced New Modified Weibull*
MO-FEW	*: Marshall-Olkin flexible Weibull extension*
NFW	*: New Flexible Weibull*
VFW	*: Very Flexible Weibull*
EIFW	*: Exponentiated Inverse Flexible Weibull*
APT	*: Alpha Power Transformation*
APW	*: Alpha Power Weibull*
APTW	*: Alpha Power Transformed Weibull*
APWQ	*: Alpha Power Within Weibull Quantile*
WEE	*: Weibull Exponentiated Exponential*
WW	*: Weibull Weibull*
EW	*: Exponentiated Weibull*
BW	*: Beta Weibull*
BMW	*: Beta Modified Weibull*
MBW	*: Modified Beta Weibull*
GMW	*: Generalized Modified Weibull*
GAPW	*: Gull Alpha Power Weibull*

1 Introduction

The Weibull [1] distribution which is named after the Swedish mathematician Waloddi Weibull is a very popular lifetime distribution in reliability theory. It is one of the most popular distributions in analyzing the lifetime data due to the wide variety of shapes it can assume by varying its parameters. It is commonly used for analyzing biological, medical, and hydrological data sets. In analyzing lifetime data one may often use the Weibull distribution. The Weibull distribution is one of the most commonly used distributions and has wide applications in the broad field of statistics. The Weibull distribution has several desirable properties which enable it to be used frequently. It is well known that the Weibull probability density function can be decreasing or unimodal, and hazard function can be either decreasing or increasing depending on the shape parameters. On modeling the monotonic hazard rates, one may prefer to use the Exponential, Gamma and Weibull distribution over other distributions. However, in case of non-monotonic hazard rates such as the bathtub-shaped hazard rates or upside down bathtub-shaped hazard rates, these distributions are not realistic or reasonable. The idea of developing new distributions by adding one or more extra parameter to an existing family of distribution functions has become important in the field of statistical distribution research as introducing an extra parameter brings more flexibility to the class of distribution functions, its usefulness and in incorporating a family of distributions [2]. Moreover, it is found more useful for data analysis purposes.

Over the years many generalization of the Weibull distribution have been studied by various researchers, the exponentiated Weibull distribution (Mudholkar & Srivastava [3]; Pal et al. [4];Nassar et al. [5];Mustafa et al. [6]; Abdullah et al. [7]), a review has been presented by Nadarajah et al.,[8] on exponentiated Weibull distribution, extended Weibull distribution (Marshall & Olkin [9]), modified Weibull distribution (Xie et al. [10]; Lai et al. [11]; Sarhan et al. [12]; Doostmoradi et al. [13]; Almalki [14]), flexible Weibull extension (Bebbington et al. [15]; Mustafa et al. [6]; Mustafa et al. [16]; Ahmad et al. [17]; Ahmad et al. [18]; park et al. [19]) and the transformed Weibull distribution (Alzaatreh et al. [20]). The hazard rate function of this distribution can be increasing, decreasing, bathtub shaped or unimodal. The recent time development in the theory and application of Weibull distribution is the Alpha Power Transformation proposed by Mahdavi & Kundu [21], where a parameter α is introduced to bring more flexibility and to incorporate skewness to the family of distribution. Nassar et al. [22] reported a new lifetime model called Alpha Power Weibull (APW) distribution with its properties and applications. Dey et al. [23] proposed a three parameter Weibull distribution referred to as Alpha Power Transformed Weibull (APTW) distribution with application and Nassar et al. [24] proposed a new family of generalized distributions based on Alpha Power Transformation. Recently Elbatal et al. [25] proposed a new technique of APT method namely New Alpha Power Weibull (NAPW) distribution and the proposed distribution offers greater flexibility. Many generalization of the Weibull distribution have been investigated by numerous researchers by increasing the number of parameters. Recently Ijaz et al. [26], proposed a new family of distribution called Gull Alpha Power Family of distribution. The special case of this family is derived by employing the Weibull distribution called Gull Alpha Power Weibull distribution (GAPW). The proposed distribution will not only model the monotonic and non-monotonic hazard rate function, but also increase flexibility and provided a better fit as compared to other probability distribution distributions provided in the literature.

It is shown that those distributions with three or more parameters are more flexible and exhibit bathtub shaped hazard rates. Bathtub shape failure rate functions are important in reliability theory, such as human life period and electronic devices life period [27]. A study on modeling of bathtub shape hazard rate function is conducted by Wang et al. [28]. The main aim of this paper is to provide an overview of the recent development of Weibull models and to study the Alpha power transformation (APT) methods. The rest of the paper is organized as follows, Section 1 provides the introduction of Weibull models and its various generalization, Section 2 provides some recent development in Weibull models, Section 3 provides the methodology and Section 4 concludes the paper with future work.

2 Recent Development in Weibull Models

The traditional Weibull distribution proposed by Weibull [1], which includes the Exponential and Rayleigh distributions as particular cases, is one of the important lifetime distributions. Even though it is one of the popular model for analyzing lifetime data, the general two parameter Weibull distribution is unable to capture the behaviour of the lifetime data sets with non-monotonic hazard rates h(t). This has led the researchers to generalize the Weibull distribution by adding additional parameters. Modifications of the Weibull distributions have been discussed in Murthy et al. [29]. Pham & Lai [27] and Lai et al. [30] provides a review on Weibull distribution. We now discuss some of the recent studies in the field of Weibull distribution that have taken place since 2011.

The reliability, hazard rate and the cumulative hazard rate function for a continuous random variable $t > 0$ is given by,

$$R(t) = 1 - F(t) \tag{2.1}$$

$$h(t) = \frac{f(t)}{R(t)} \tag{2.2}$$

$$H(t) = \int_0^t h(x)dx \tag{2.3}$$

The shape of the hazard rate function depends only on the shape parameters. The three possible shapes are mention below:

(i) If the shape parameter is less than 1, then h(t) is decreasing.
(ii) If the shape parameter is greater than 1, then h(t) is increasing.
(iii) If the shape parameter is equal to 1, then h(t) is constant.

and

$$H(t) = \int_0^t h(x)dx \tag{2.3}$$

Where the cumulative hazard rate function H(t) is non-negative and non-decreasing for all $t \geq 0$, and it must satisfy the following conditions:

1) $H(t) = 0$
2) $\lim_{t\to\infty} H(t) = \infty$

2.1 Weibull Distribution Extensions

Over the years various researchers have contributed various extensions that are either derived or related to the basic Weibull distribution model. Here we will be discussing recent extensions of Weibull distribution along with the distributions in which the shapes of the density function and the hazard function have been studied.

2.1.1 Exponentiated generalised flexible Weibull Extension distribution

The reliability function R(t) of the four parameter Exponentiated Generalised Flexible Weibull Extension (EG-FWE) distribution [6] is given as,

$$R(t) = 1 - \left[1 - exp\left\{-ae^{\alpha t - \frac{\beta}{t}}\right\}\right]^b;\ t > 0,\ a, b, \alpha, \beta > 0$$

and the corresponding hazard rate function h(t) is given as,

$$h(t) = \frac{ab\left(\alpha + \frac{\beta}{t^2}\right)exp\left\{\alpha t - \frac{\beta}{t} - ae^{\alpha t - \frac{\beta}{t}}\right\}\left[1 - exp\left\{-ae^{\alpha t - \frac{\beta}{t}}\right\}\right]^{b-1}}{1 - \left[1 - exp\left\{-ae^{\alpha t - \frac{\beta}{t}}\right\}\right]^b};\ t > 0, a, b, \alpha, \beta > 0$$

with $a > 0$ and $b > 0$ are two additional parameters.

2.1.1.1 Shape behaviour of EG-FEW distribution

IFR if $a > 1, b > 1$
DFR if $a < 1, b \geq 1$

This distribution has increasing and decreasing failure rate.

2.1.2 Exponentiated inverse flexible Weibull distribution

The reliability function of Exponentiated Inverse Flexible Weibull (EIFW) extension distribution [31] is given as,

$$R(t) = 1 - e^{-\lambda e^{\frac{\alpha}{t} - \beta t}};\ t > 0, \alpha, \beta, \lambda > 0$$

and the corresponding hazard rate function is given as,

$$h(t) = \frac{\lambda\left(\beta + \frac{\alpha}{t^2}\right)e^{\frac{\alpha}{t} - \beta t}e^{-\lambda e^{\frac{\alpha}{t} - \beta t}}}{1 - e^{-\lambda e^{\frac{\alpha}{t} - \beta t}}};\ t > 0, \alpha, \beta, \lambda > 0$$

2.1.2.1 Shape behavior of EIFW distribution

The failure rate function of the EIFW extension distribution can take different shapes based on the values of α, β and, which makes the EIFW model more flexible to fit different lifetime data sets.

2.1.3 Weibull exponentiated Weibull distribution

The reliability function of the Weibull Exponentiated Weibull (WEW) model [7] of continuous distribution is given as,

$$R(t) = exp\left(\frac{-\rho\left[1 - e^{-(\delta t)^{\epsilon}}\right]^{\vartheta\sigma}}{\{1 - [1 - e^{-(\delta t)^{\epsilon}}]^{\vartheta}\}^{\sigma}}\right) ; t > 0, \delta, \epsilon, \vartheta, \rho, \sigma > 0$$

and the corresponding hazard rate function h(t) is given as,

$$h(t) = \rho\sigma\epsilon\vartheta\delta^{\epsilon}t^{\epsilon-1}e^{-(\delta t)^{\epsilon}}\frac{\left[1 - e^{-(\delta t)^{\epsilon}}\right]^{\vartheta\,\sigma-1}}{\{1 - [1 - e^{-(\delta t)^{\epsilon}}]^{\vartheta}\}^{\sigma+1}};\ t > 0, \delta, \epsilon, \vartheta, \rho, \sigma > 0$$

where $\rho > 0$ and $\sigma > 0$ is the two additional parameters.

2.1.3.1 Shape behaviour of WEW distribution

IFR if $\delta > 1, \vartheta < 1, \epsilon > 1$
BT if $\delta > 1, \vartheta \geq 1, \epsilon > 1$
DFR if $\delta < 1, \vartheta > 1, \epsilon < 1$

This distribution has more flexible shapes like increasing, decreasing and bathtub-shape and this distribution is more flexible than any other sub-models like WEE, WW and EW.

2.1.4 New modified Weibull distribution

The reliability function of the New Modified Weibull (NMW) distribution [13] is given as,

$$R(t) = e^{-e^{\alpha t^{\gamma}} + e^{-\beta t^{\lambda}}} ; t > 0, \alpha, \beta > 0, \gamma, \lambda \geq 0$$

and the corresponding hazard rate function h(t) is given as,

$$h(t) = \alpha\gamma t^{\gamma-1}e^{\alpha t^{\gamma}} + \lambda\beta t^{\lambda-1}e^{-\beta t^{\lambda}}; t > 0, \alpha, \beta > 0, \gamma, \lambda \geq 0$$

where $\alpha, \beta > 0$ is the scale parameters and $\gamma, \lambda \geq 0$ is the shape parameters.

2.1.4.1 Shape behaviour of NMW distribution

DFR if $\alpha, \beta, \gamma, \lambda < 1$
IFR if $\alpha > 1$ and $\beta, \gamma, \lambda = 1$
BT if $\alpha, \beta, \gamma, \lambda$ <1
UFR if $\alpha, \beta, \gamma < 1$ and $\lambda = 2$
IDI if $\alpha, \beta, \gamma < 1$ and $\lambda = 2.5$
DID if $\alpha < 1$ and $\beta, \gamma, \lambda > 1$

This model is more flexible as compared to MB, EW, MW, BW, BMW and GMW distribution. The proposed distribution function has decreasing, unimodal and bimodal pdf. Also, the NMW distribution has more general form of failure rate function.

2.1.5 Reduced New Modified Weibull distribution

The reliability function of the reduced new modified Weibull (RNMW) distribution [14] is given as,

$$R(t) = e^{-\alpha\sqrt{t} - \beta\sqrt{t}e^{\lambda t}} ; t > 0, \alpha, \beta, \lambda > 0$$

and the corresponding hazard rate function h(t) is given as,

$$h(t) = \frac{1}{2\sqrt{t}}\left[\alpha + \beta(1 + 2\lambda t)e^{\lambda t}\right]; t > 0, \alpha, \beta, \lambda > 0$$

where $\alpha, \beta > 0$ is the scale parameters and $\lambda > 0$ is the shape parameter.

2.1.5.1 Shape behaviour of RNMW distribution

This distribution has bathtub shape hazard rate function. It is a more simplified distribution with originally five parameters reduced to three parameters.

2.1.6 Marshall-Olkin Flexible Weibull Extension distribution

The reliability of the three parameter Marshall-Olkin flexible Weibull extension (MO-FWE) distribution [16] is given as,

$$R(t) = \frac{\theta e^{-e^{\alpha t - \frac{\beta}{t}}}}{1-(1-\theta)e^{-e^{\alpha t - \frac{\beta}{t}}}} \; ; t > 0, \alpha, \beta, \theta > 0$$

and the corresponding hazard rate function h(t) is given as,

$$h(t) = \frac{(\alpha + \frac{\beta}{t^2})e^{\alpha t - \frac{\beta}{t}}}{1 - (1-\theta)e^{-e^{\alpha t - \frac{\beta}{t}}}}; t > 0, \alpha, \beta, \theta > 0$$

where $\theta > 0$ is the additional parameter.

2.1.6.1 Shape behaviour of MO-FEW distribution

IFR if $\alpha > 1, \beta > 1, \theta < 1$
BT if $\alpha < 1, \beta < 1, \theta < 1$

This distribution has either increasing or bathtub shape hazard rate functions.

2.1.7 New flexible Weibull distribution

The reliability function of the new flexible Weibull (NFW) distribution [18] is given as,

$$R(t) = e^{-e(\beta t^{\gamma} + \theta t)}; t > 0, \gamma, \beta, \theta > 0$$

and the corresponding hazard rate function h(t) is given as,

$$h(t) = (\gamma \beta t^{\gamma - 1} + \theta)e^{(\beta t^{\gamma} + \theta t)}; t > 0, \gamma, \beta, \theta > 0$$

2.1.7.1 Shape behaviour of NFW distribution

IFR if $\theta > 1, \beta < 1$ and with $\gamma = 1$
BT if $\theta < 1, \beta < 1, \gamma < 1$

This distribution can be used for modeling increasing and bathtub shape hazard rate functions.

2.1.8 New extended flexible Weibull distribution

The reliability function of the New Extended Flexible Weibull (NEx-FW) distribution [17] is given as,

$$R(t) = e^{-e^{(\beta t^{\gamma}+\theta t^2)}};\ \ t > 0, \gamma, \beta, \theta > 0$$

and the corresponding hazard rate function is given as,

$$\square(t) = (\gamma\beta t^{\gamma-1} + 2\theta t)e^{(\beta t^{\gamma}+\theta t^2)};\ t > 0, \gamma, \beta, \theta > 0$$

2.1.8.1 Shape behaviour of NEx-FW distribution

BT shape hazard rate function, if $\beta > 1$, $\theta < 1$ and $\gamma < 1$

2.1.9 Very flexible Weibull distribution

The reliability function of the very flexible Weibull (VFW) distribution [19] is

$$\square(t) = exp\left\{-e^{-\mu+\alpha t}\left(log(t+1)^{\beta}\right)\right\}$$

and the corresponding hazard rate function h(t) is given as,

$$\square(t) = exp(\mu + \alpha t) + \left(\alpha log(t+1) + \frac{\beta}{t+1}\right)\left(log(t+1)\right)^{\beta-1}$$

where $\alpha > 0$ is the scale parameter and $\beta > 0, \mu > 0$ is the shape parameter.

2.1.9.1 Shape Behaviour of VFW distribution

IFR if $\alpha \leq 1, \beta < 1$ and $\mu = 0$
DFR if $\alpha < 1, \beta < 1$ and $\mu = 0$
BT, MBT and UBT if $\alpha < 1, \beta > 1$ and $\mu = 0$

2.2 New Extension of Weibull distribution

The popularity of Weibull distribution in the studies of reliability theory has led the researchers for a new extension of Weibull distribution.

2.2.1 Alpha power Weibull distribution

The reliability function of the alpha power Weibull (APW) distribution [22] is given as,

$$R(t) = \begin{cases} \frac{\alpha}{\alpha-1}\left(1 - \alpha^{-e^{-\lambda t^{\beta}}}\right); t > 0, \alpha > 0, \alpha \neq 1 \\ 1 - e^{-\lambda t^{\beta}} \qquad ; t > 0, \alpha = 1 \end{cases}$$

and the corresponding hazard rate function h(t) is given as,

$$\square((t) = \begin{cases} \log(\alpha)\,\lambda\beta t^{\beta-1}e^{-\lambda t^{\beta}}\left(\alpha^{e^{-\lambda t^{\beta}}} - 1\right)^{-1}; t > 0, \alpha > 0, \alpha \neq 1 \\ \lambda\beta t^{\beta-1} \qquad ;\ t > 0, \alpha = 1 \end{cases}$$

where $\alpha > 0$ and $\beta > 0$ is the shape and $\lambda > 0$ is the scale parameter.

2.2.1.1 Shape Behaviour of APW distribution

DFR if $\alpha < 1, \beta \leq 1$
IFR or concave-convex shape if $\alpha < 1, \beta > 1$
DFR or concave-convex shape if $\alpha > 1, \beta < 1$

2.2.2 Alpha Power transformed Weibull distribution

The reliability function of the alpha power transformed Weibull (APTW) distribution [23] is given as,

$$R(t) = \begin{cases} \dfrac{\alpha}{\alpha - 1}\left(1 - \alpha^{-e^{-\lambda t^{\beta}}}\right); & \alpha > 0, \alpha \neq 1 \\ e^{-\lambda t^{\beta}} & ; \quad \alpha = 1 \end{cases}$$

and the corresponding hazard rate function h(t) is given as,

$$h(t) = \begin{cases} \dfrac{\log(\alpha)\, \beta\lambda t^{\lambda-1} e^{-\beta t^{\lambda}} \alpha^{-e^{-\beta t^{\lambda}}}}{1 - \alpha^{-e^{-\beta t^{\lambda}}}}; & t > 0, \alpha, \beta, \lambda > 0, \alpha \neq 1 \\ \beta\lambda t^{\lambda-1} & ; \ t > 0, \alpha, \beta, \lambda > 0, \alpha \neq 1 \end{cases}$$

where $\alpha > 0$ and $\beta > 0$ is the shape and $\lambda > 0$ is the scale parameter. This model is capable of modelling non-monotonic hazard rates, bathtub, upside-down and increasing-decreasing- increasing hazard rates. It can be viewed as a suitable model for fitting the skewed data which may not be commonly fitted by other known distributions.

2.2.2.1 Shape Behaviour of APTW distribution

Constant if $\alpha = 1, \beta = 1, \lambda = 1$
IFR if $\alpha > 1, \beta = 1, \lambda = 1$
DFR if $\alpha < 1, \beta \geq 1, \lambda \leq 1$
IDI if $\alpha < 1, \beta = 1, \lambda > 1$
BT if $\alpha > 1, \beta = 1, \lambda < 1$

2.2.3 Alpha power within Weibull quantile distribution

The reliability function of the alpha power within Weibull quantile (APWQ) distribution [24] is given as,

$$R(t) = \frac{\log(\alpha) - \log\left(1 + (\alpha - 1)\left(1 - e^{-\lambda t^{\beta}}\right)\right)}{\log(\alpha)}\ ; t > 0, \alpha > 0, \alpha \neq 1$$

and the corresponding hazard rate function h(t) is given as,

$$h(t) = \frac{(\alpha - 1)\lambda\beta t^{\beta-1} e^{-\lambda t^{\beta}}}{\left(1 + (\alpha - 1)\left(1 - e^{-\lambda t^{\beta}}\right)\right)\left[\log(\alpha) - \log\left(1 + (\alpha - 1)\left(1 - e^{-\lambda t^{\beta}}\right)\right)\right]};\ t > 0, \alpha > 0, \alpha \neq 1$$

where $\alpha > 0$ and $\beta > 0$ is the shape and $\lambda > 0$ is the scale parameter. It is observed that in terms of shape APWQ is more flexible.

2.2.3.1 Shape behaviour of APWQ distribution

IFR if $\alpha < 1, \beta > 1$
DFR if $\alpha < 1, \beta < 1$
BT or UBT or Bimodal if $\alpha > 1, \beta > 1$

2.2.4 New alpha power transformed Weibull distribution

The reliability function of the new alpha power transformed Weibull (NAPTW) distribution [25] is given as,

$$R(t) = \frac{\alpha - (1 - e^{-\gamma t^{\theta}})\alpha^{(1-e^{-\gamma t^{\theta}})}}{\alpha};\ t > 0, \alpha > 0, \alpha \neq 1$$

and the corresponding hazard rate function h(t) is given as,

$$h(t) = \frac{\gamma\theta t^{\theta-1}e^{-\gamma t^{\theta}}}{\alpha - \left(1 - e^{-\gamma t^{\theta}}\right)}\left[1 + \log(\alpha)\left(1 - e^{-\gamma t^{\theta}}\right)\right];\ t > 0, \alpha > 0, \alpha \neq 1$$

where $\alpha > 0$ and $\beta > 0$ is the shape and $\lambda > 0$ is the scale parameter. For different value of α, γ and θ the NAPW distribution reduces to several sub models such as Weibull distribution, Exponential distribution, Rayleigh distribution, NAPT Exponential distribution and NAPT Rayleigh distribution.

2.2.4.1 Shape Behaviour of NAPW distribution

Symmetric shape if $\alpha > 1, \theta > 1, \gamma > 1$
Negatively skewed if $\alpha > 1, \theta > 1, \gamma < 1$
Positively skewed if $\alpha > 1, \theta > 1, \gamma > 1$

Hazard rate function has the following shapes:

Bimodal shape if $\alpha = 1, \theta = 1, \gamma > 1$
Monotonically increasing if $\alpha > 1, \theta > 1, \gamma > 1$
Monotonically decreasing if $\alpha < 1, \theta < 1, \gamma > 1$

The alpha power extension models are one of the recent developments in the field of Weibull distribution. It provides several desirable properties and it is more flexible. It is also useful for incorporating skewness to a family of distribution. Also, the alpha power extension model provides several sub-models for different values of the parameter.

2.2.5 Gull alpha power Weibull distribution

The reliability function of the Gull Alpha Power Weibull distribution (GAPW) distribution [27] is given as,

$$R(t) = 1 - \alpha^{e^{-\beta t^{\gamma}}}\left(1 - e^{-\beta t^{\gamma}}\right); t > 0, \alpha, \beta, \gamma > 0$$

And the corresponding hazard rate function is given as,

$$h(t) = \alpha^{e^{-\beta t^{\gamma}}}\beta\gamma t^{\gamma-1}e^{-\beta t^{\gamma}}\left[\frac{1 - \alpha^{e^{-\beta t^{\gamma}}}\left(1 - e^{-\beta t^{\gamma}}\right) - \log(\alpha) - e^{-\beta t^{\gamma}}\log(\alpha)}{1 - \alpha^{e^{-\beta t^{\gamma}}}(1 - e^{-\beta t^{\gamma}})}\right]$$

The proposed distribution contains three parameters that is $\beta>0$ is the scale and $\alpha>0$, $\gamma>0$ being the shape parameters.

2.2.5.1 Shape behaviour of GAPW distribution

Decreasing function if $\alpha > 1$, $\beta < 1$ and $\gamma < 1$
Increasing decreasing function if $\alpha > 1$, $\beta > 1$ and $\gamma > 1$

3 Methodology

3.1 Parameter Estimation

The different methods that are used for parameter estimation for the Weibull distribution are:-

(a) Method of Moments
(b) Method of Maximum Likelihood
(c) Bayesian method

3.2 Mathematical Properties

The important mathematical properties discussed for the above Weibull distribution are as follows:

(a) Quantile function
(b) Order statistics
(c) Moments
(d) Mean residual life and mean waiting time
(e) Median
(f) Mode
(g) Stress-Strength parameter
(h) Rényi and Shannon entropies
(i) Stochastic Ordering
(j) Bonferroni and Lorenz Curve

3.3 Modeling of Data

To check whether a given data sets can be modeled by one or various extensions of the Weibull models, the following three modeling steps are considered:

Step 1: Model Selection
Step 2: Estimation of model parameters
Step 3: Model Validation or Goodness of fit test

4 Conclusion and Future Work

In this article we present various extensions of Weibull models and discuss its various characteristics and shape behaviors. In this study, we observed that these extensions of Weibull models are more appropriate for modeling complex data sets because of the many different shapes of the reliability and hazard rate function. We also observed that as we increase the number of parameters it tends to bring more flexibility in the distribution. We then present the parameter estimation methods, mathematical properties and modeling of data. In recent past, Mahadavi & Kundu [21] introduced a new transformation method known as the Alpha Power Transformation (APT) method where a new parameter $'\alpha'$ is introduced [23, 25, 21, 24, 22]. Several researchers have provided various extensions of APT method by using several distributions. Particularly, the researchers are more interested to study APT method on Weibull distribution. The APT models offers greater distributional flexibility and is able to model lifetime data with monotonic, non-monotonic, constant and bathtub-shaped hazard rate function. Also, by analyzing the simulation behavior of the various extension of Weibull distribution, we observed that the mean square error (MSE) and bias, for different values of the parameters decreases as the sample size n increases showing the reliability of the APT method on Weibull distribution. Recently, Dey et al. [23] provides a brief explanation about the bivariate APT method on Weibull distribution. Thus, a detailed study about the bivariate APT method with properties and application will be a great field to explore in future.

Acknowledgement

The authors would like to thank the reviewers for their valuable comments and suggestions. It really enhances the quality of the paper.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Weibull W. A statistical distribution function of wide applicability. Journal of Applied Mechanics. 1951;51:293-297.

[2] Lee C, Famoye F, Alzaatreh AY. Methods for generating families of univariate continuous distributions in the recent decades. Wiley Interdisciplinary Reviews: Computational Statistics. 2013;5:219-238. DOI: 10.1002/wics.1255.

[3] Mudholkar GS, Srivastava DK. Exponentiated Weibull family for analyzing bathtub failure rate data. IEEE Transactions on Reliability. 1993;42(2):299-302.

[4] Pal M, Ali MM, Woo J. Exponentiated Weibull distribution. Statistica. 2003;66(2):141-147.

[5] Nassar MM, Eissa FH. On the exponentiated Weibull distribution. Communication in Statistics- Theory and Methods. 2003;32(7):1317-1333.

[6] Mustafa A, El-Desouky BS, Al-Garash S. The exponentiated generalized flexible Weibull extension distribution. Fundamental Journal of Mathematics and Mathematical Sciences. 2016;6(2):75-98.

[7] Abdullah M, Ibrahim NA. New extension of exponentiated Weibull distribution with properties and application to survival data. Proceeding Second ISI Regional Statistics Conference. 20-24 March, Indonesia (Session CPS34); 2017.

[8] Nadarajah S, Cordeiro GM, Ortega EMM. The exponentiated Weibull distribution: a survey. Springer. 2013;54:839-877.

[9] Marshall AW, Olkin I. A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families. Biometrica. 1997;84(3), 641-652.

[10] Xie M, Tang Y, Goh TN. A modified extension with bathtub-shaped failure rate function. Reliability Engineering and System Safety. 2002;76(3):279-285.

[11] Lai CD, Xie M. A modified Weibull distribution. IEEE Transaction on Reliability. 2009;52(1):33-37.

[12] Sarhan AM, Zaindin M. Modified Weibull distribution. Applied Sciences. 2009;11:123-136.

[13] Doostmoradi A, Zadkarami MR, Sheykhabad AR. A new modified Weibull distribution and its application. Journal of Statistical Research of Iran. 2014;11:97-118.

[14] Almalki SJ. A reduced new modified Weibull distribution. Communication in Statistics- Theory and Methods. 2018;47(10):2297-2313.

[15] Bebbington M, Lai CD, Zitikis R. A flexible Weibull extension. Reliability Engineering and System Safety. 2007;92:719-726.

[16] Mustafa A, El-Desouky BS, Al-Garash S. The marshall-olkin flexible Weibull extension distribution. 2016;arXiv:1609.08997v1 [math.ST]

[17] Ahmad Z, Hussain Z. The new extended flexible Weibull distribution and its applications. International Journal of Data Science and Analysis. 2017;3(3):18-23. DOI: 10.11648/j.ijdsa.20170303.11

[18] Ahmad Z, Hussain Z. New flexible Weibull distribution. International Journal of Advanced Research in Electrical, Electronics and Instrumentation Engineering. (An ISO 3297: 2007 Certified Organization), 2017;6(5).

[19] Park S, Park J. A general class of flexible Weibull distributions. Communication in Statistics-Theory and Methods. 2018;47(4):767-778.

[20] Alzaatreh A, Lee C, Famoye F. A new method of generating families of continuous distributions. Metron. 2014;71:63-79, (arXiv:1609.08997v1 [math.ST]

[21] Mahdavi A, Kundu D. A new method for generating distributions with an application to Exponential distribution. Communication in Statistics-Theory and Methods. 2017;46(13):6543-6557.

[22] Nassar M, Alzaatreh A, Mead M, Abo-Kasem O. Alpha power Weibull distribution: properties and applications. Communication in Statistics-Theory and Methods. 2017;46(20):10236-10252.

[23] Dey S, Sharma VK, Mesfioui M. A new extension of Weibull distribution with application to lifetime data. Annals of Data Science. 2017;4(1):31–61.

[24] Nassar M, Alzaatreh A, Abo-Kasem O, Mead M, Mansoor M. A new method of generalized distributions based on alpha power transformation with application to cancer data. Annals of Data Science; 2018.

[25] Elbatal I, Ahmad Z, Elgarhy M, Almarashi AM. A new alpha power transformed family of distributions: properties and applications to the Weibull model. Journal of Nonlinear Sciences and Applications. 2019;12:1-20.

[26] Ijaz M, Asim SM, Alamgir Farooq, Khan M, Manzoor SA,S. A gull alpha power Weibull distribution with applications to real and simulated data. PLOS ONE. 2020;15(6):. DOI:https://doi.org/10.1371/journal.pone.0233080

[27] Pham H, Lai CD. On recent generalization of the Weibull distribution. IEEE Transactions on Reliability. 2007;56(3):454-458.

[28] Wang KS, Hsu FS, Liu PP. Modeling the bathtub shape hazard rate function in terms of reliability. Reliability Engineering and System Safety. 2002;75:397-406.

[29] Murthy DNP, Xie M, Jiang R. Weibull models. John Wiley and Sons, New York; 2003.

[30] Lai CD, Murthy DNP, Xie M. Weibull distributions. John Wiley and Sons. 2011;282-287.

[31] El-Morshedy M, El-Bassiouny AH, El-Gohary A. Exponentiated inverse flexible weibull extension distribution. Journal of Statistics Applications & Probability. 2017;6 (1):169-183.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://www.sdiarticle4.com/review-history/69405