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A New Algorithm for Approximate Maximum Likelihood Estimation in Sub-fractional Chan-Karolyi-Longstaff-Sanders Model

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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Abstract

The paper introduces several approximate maximum likelihood estimators of the parameters of the sub-fractional Chan-Karolyi-Longstaff-Sanders (CKLS) interest rate model and obtains their rates of convergence. A new algorithm inspired by Newton-Cotes formula is presented to improve the accuracy of estimation. The estimators are useful for simulation of interest rates. The proposed new algorithm could be useful for other stochastic computation. It also proposes a generalization of the CKLS interest rate model with sub-fractional Brownian motion drivers which preserves medium range memory.

Keywords: Itô stochastic differential equation; sub-fractional Brownian motion; sub-fractional diffusion process; discrete observations; term structure of interest rates; approximate maximum likelihood estimators; Newton-Cotes distribution.

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1 Introduction

Factors and parsimony are two important objects in a financial model. The model should have sufficient factors describing the behavior of the market and the model should be parsimonious, i.e., should have enough parameters describing the properties of the model.

High-frequency and ultra high-frequency data analysis in finance is the recent trend of investigation, see Ait-Sahalia and Jacod [1]. We introduce some new approximate maximum likelihood estimators of the drift parameters in the Chan-Karolyi-Longstaff-Sanders model (CKLS model hereafter), introduced by Chan et al. [2], based on discretely sampled interest rate data and propose a new algorithm to obtain faster rates of convergence of the estimators to the corresponding continuous maximum likelihood estimators.

2 Term Structure Models and Derivative Pricing

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a stochastic basis on which is defined the is a sub-fractional Brownian motion (sub-FBM) W_t^H which is a centered Gaussian process with covariance function

$$C_H(s,t) = s^{2H} + t^{2H} - \frac{1}{2}\left[(s+t)^{2H} + |s-t|^{2H}\right], \quad s,t>0$$

for $0 < H < 1$ introduced by Bojdecki, Gorostiza and Talarczyk [3]. The interesting feature of sub-FBM is that this process has some of the main properties of FBM, but the increments of the process are nonstationary, more weakly correlated on non-overlapping time intervals than that of FBM, and its covariance decays polynomially at a higher rate as the distance between the intervals tends to infinity. The parameter H governs the memory behavior of the model. For $H = 0.5$, this process reduces to the standard Brownian motion with short memory. We assume that $H > 0.5$ for which the process has medium range long memory.

Recall that a fractional Brownian motion (FBM) has the covariance

$$\widetilde{C}_H(s,t) = \frac{1}{2}\left[s^{2H} + t^{2H} - |s-t|^{2H}\right], \quad s,t>0.$$

For $H > 0.5$ the process has long rande dependence or long memory.

A real asset price model should be of the following hybrid type with 14 parameters. We consider the hybrid stochastic volatility, stochastic interest rate, stochastic leverage and stochastic elasticity model under the risk neutral measure which is given by

$$\begin{aligned}
dS_t &= X_t dt + \sqrt{V_{t-}} S_t dW_t + \rho_{\lambda t} dL_{\tau_{\lambda t}},\\
dV_t &= -\lambda V_t dt + \upsilon_{\lambda t-} dL_{\tau_{\lambda t}},\\
dX_t &= \alpha(\beta - X_t)dt + \sigma X_t^{\gamma_t} dW_t^H,\\
d\rho_t &= ((2\zeta - \eta) - \eta\rho_t)dt + \theta\sqrt{(1+\rho_t)(1-\rho_t)}dZ_t,\\
d\xi_t &= \kappa(\mu - \xi_t)dt + \varsigma\sqrt{\xi_t}dB_t.\\
d\gamma_t &= \varpi(\psi - \delta))dt + \sqrt{\chi}dM_t.\\
d\tau_t &= \xi_{t-}dt.
\end{aligned}$$

where L_t is a Levy process, W^H is a subfractional Brownian motion, B_t, Z_t and M_t are standard Brownian motions. Here S_t is the asset price which a geometric jump-diffusion, V_t is the stochastic volatility which is a Levy O-U process, X_t is the stochastic interest rate which is a sub-fractional

Chan-Karolyi-Longstaff-Sanders (CKLS) process, ρ_t is the stochastic leverage which is a Jacobi (Beta) process, ξ_t is a volatility modulation (stochastic time change) of the driving Levy subordinator which is a Cox-Ingersoll-Ross (CIR) process, γ_t is the stochastic elasticity models which is another CIR process, and all the 14 parameters $\lambda, \alpha, \beta, \sigma, \xi, \eta, \theta, \kappa, \mu, \varsigma, \varpi, \psi, \delta, \chi$ are positive.

Recently Filipovic et al. [4] studied linear-rational term structure models with martingale error terms which ensures nonnegative interest rates, accommodates unspanned factors affecting volatility and risk premiums, and admits semi-analytical solutions to swaptions-an important class of interest rate derivatives, that underlie the pricing and hedging of mortgage-backed securities, callable agency securities, life insurance products, and a wide variety of structured products. As shown by Filipovic et al. [4] a parsimonious model specification within the linear rational class has a very good fit to both interest rate swaps and swaptions from 1997 to 2003 and captures many features of term structure, volatility, and risk premium dynamics-including when interest rates are close to the zero lower bound. The term structure is assumed to be driven by three factors.

Polynomial diffusions, studied in Filipovic and Larsson [5], represent an extension of affine class which have linear drift and quadratic diffusion functions. A state price density is a positive semimartingale $\zeta_t = e^{-\alpha t}p(X_t)$ where p is a positive polynomial of a factor diffusion process X defined on a filtered probability space. The cash-flow $C_T = q(X_T)$ for some polynomial q. The fact that pq itself being a polynomial implies that the model price $\Pi(t,T) = \frac{1}{\zeta_t}E[\zeta_T C_T|\mathcal{F}_t]$ becomes a rational function of X_t with coefficients given in terms of matrix exponential. Polynomial diffusion models thus yield closed form expressions for any security with cash flows specified as a polynomial function of X, which makes them universally applicable in finance. This includes financial market models for interest rates (with $C_T = 1$), credit risk in a doubly stochastic framework (with C_T the conditional survival probability), stochastic volatility (with C_T the spot variance), and commodities and electricity (with C_T the spot price). They showed uniqueness of polynomial diffusions via moment determinancy in the classical theory of moment problems in combination with pathwise uniqueness. Mixed moments of all finite-dimensional marginal distributions of a polynomial diffusions are uniquely determined by its generator, uniqueness follows whenever these moments determine the underlying distribution. This is often true, for instance in the affine case or when the state space is compact, or more generally if exponential moments exist. There are however situations exists where the moment problem fails. In that case Yamada-Watanabe type arguments (see Ikeda and Watanabe [6]) are used which give uniqueness in one dimensional case as well as when the process dynamics exhibits a certain hierarchical structure. These uniqueness results do not depend on the geometry of the state space. Existence boils down to a stochastic invariance problem that is solved for semialgebraic state spaces.

Zhou [7] used one dimensional polynomial jump diffusions to build short rate models that were estimated by data using generalized method of moments approach. Examples of non-afine polynomial processes include multidimensional Jacobi or Fisher-Wright processes (Ethier [8], Pearson diffusions (Forman and Sørensen [9]), Dunkl processes (Dunkl [10]), Gallardo and Yor [11]). A short rate model based on the Jacobi process was presented by Delbaen and Shirakawa [12] and Larsen and Sørensen [13] used the same process for exchange rate modeling. Forman and Sørensen [9] studied parameter estimation in non-affine polynomial diffusions.

Sub-fractional polynomial jump-diffusion models which nest affine sub-fractional jump-diffusions. Non-fractional polynomial jump-diffusion models are studied in Filipovic and Larsson [14]. A sub-fractional jump- diffusion model is polynomial if its extended generator maps any polynomial to a polynomial of equal or lower degree. As a consequence, its conditional moments can be computed in closed form. This property renders polynomial jump-diffusions computationally tractable and perfectly suitable for financial asset pricing models. Many commonly occurring sub-fractional jump

diffusions are polynomial, for example, sub-fractional Ornstein-Uhlenbeck (OU) processes, sub-fractional square-root diffusions, sub-fractional Jacobi or Wright-Fisher diffusions, Levy processes and geometric Levy processes. Subject to some integrability conditions on the jumps, sub-fractional affine jumps diffusions are polynomials but the converse is not true. Thus sub-fractional polynomial jump diffusions truely extend the class of sub-fractional affine jump-diffusions. Affine property of sub-fractional jump-diffusion is not invariant under exponentiation or subordination in general. Beyond the affine class, sub-fractional polynomial jump diffusions include sub-fractional Dunkl processes and univariate diffusion volatility models, e.g, extended sub-fractional Stein-Stein model, extended sub-fractional Hull-White model.

Another example of sub-fractional polynomial diffusion is sub-fractional GARCH diffusion which satisfies

$$dX_t = \alpha(\beta - X_t)dt + \sqrt{2\kappa}X_t dW_t^H$$

with $\alpha, \beta, \kappa > 0$. For $H = 0.5$, the invariant distribution is inverse gamma distribution with shape parameter 2 and scale parameter $1/\kappa$. In the stationary case, when X_0 has the invariant distribution, we have $E(X_t) = \beta$ and $E(X_t^2) = \infty$.

Sub-fractional polynomial diffusions are invariant under exponentiation and subordination. However, affine property is not invariant under exponentiation. One can generate non-affine polynomial sub-fractional diffusions from affine sub-fractional diffusions. Consider the sub-fractional square-root process

$$dX_t = \alpha(\beta - X_t)dt + \sigma\sqrt{X_t}dW_t^H$$

which is an affine sub-fractional diffusion. The augmented process $\bar{X}_t = (X_t, X_t^2)$ satisfies

$$d\bar{X}_{1t} = (b + \beta\bar{X}_{1t})dt + \sigma\sqrt{\bar{X}_{1t}}dW_t^H,$$

$$d\bar{X}_{2t} = ((2b + \sigma^2\bar{X}_{1t}) + 2\beta\bar{X}_{2t})dt + 2\sigma\sqrt{\bar{X}_{1t}\bar{X}_{2t}}dW_t^H.$$

This shows that $\bar{X}_t = (X_t, X_t^2)$ is not affine while it is a polynomial.

One can extend the sub-fractional GARCH diffusion to have jumps driven by Poisson random measure which is a special case of the Levy driven linear SDE.

Consider the sub-fractional jump-diffusion operator $\mathcal{G}$ of the form

$$\mathcal{G}f(x) = H(2H-1)a(x)f''(x) + b(x)f'(x) + \int_{\mathbb{R}}(f(x+\xi) - f(x) - \xi f'(x))\nu(x, d\xi)$$

for $H > 0.5$

$$\mathcal{G}f(x) = Ha(x)f''(x) + b(x)f'(x) + \int_{\mathbb{R}}(f(x+\xi) - f(x) - \xi f'(x))\nu(x, d\xi)$$

for $H < 0.5$ for some measurable maps a (the volatility) and b (the drift) and transition kernel $\nu(x, d\xi)$ satisfying $\nu(x, \{0\}) = 0$ and $\int_{\mathbb{R}} |\xi| \wedge |\xi|^2\nu(x, d\xi) < \infty$.

Let X_t be a sub-fractional jump diffusion with the extended generator $\mathcal{G}$. For $H = 0.5$,

$$\mathcal{G}f(x) = \frac{1}{2}a(x)f''(x) + b(x)f'(x) + \int_{\mathbb{R}}(f(x+\xi) - f(x) - \xi f'(x))\nu(x, d\xi)$$

in which case X_t is a special semimartingale and

$$f(X_t) - f(X_0) - \int_0^t \mathcal{G}f(X_s)ds$$

is a local martingale for any bounded C^2 function $f(x)$.

Polynomial asset price models are given by financial market with m primary assets with price processes given by

$$S_{i,t} = S_{i,0} \exp\left\{ \int_0^t r_s ds + Y_{i,t} \right\}, 1 \leq i \leq m, \quad t \geq 0$$

where r_t is the risk free rate and $Y_t = (Y_{1,t}, Y_{2,t}, \ldots, Y_{m,t})$ are logarithmic excess return processes with $Y_0 = 0$.

Let X_t be a polynomial sub-fractional jump-diffusion such that $Z_t = (X_t, Y_t)$ is a sub-fractional jump diffusion with extended generator $\mathcal{G}$. We assume that

$$\int_{\mathbb{R}} (e^{\eta_i} - 1 - \eta_i) \nu^Y(x, d\eta) < \infty.$$

For $H = .5$, the price processes are special semimartingales with the decomposition

$$\frac{dS_{i,t}}{S_{i,t-}} = (r_t + \epsilon_i(X_t))dt + dY^c_{i,t} + \int_{\mathbb{R}} (e^{\eta_i} - 1)(\mu^Y(d\eta, dt) - \nu^Y(X_{t-}, d\eta)), \ i \geq 1, \ t \geq 0$$

where Y^c_t denotes the continuous martingale part of Y_t, $\mu^Y(d\eta, dt)$ is the integer-valued random measure associated with the jumps of Y_t and the simple excess rates of return are given by

$$\epsilon_i(x) = b^Y_i(x) + \frac{1}{2} a^Y_{ii}(x) + \int_{\mathbb{R}} (e^{\eta_i} - 1 - \eta_i) \nu^Y(x, d\eta).$$

The measure P is the risk-neutral measure so that the discounted price process $\exp(-\int_0^t r_s ds) S_{i,t}$ are local martingales. This is achieved by setting $\epsilon_i(x) = 0$. But $\epsilon_i(x)$ can not be zero if $Z_t = (X_t, Y_t)$ is a polynomial jump-diffusion, other than affine in general. But Z_t can be embedded into a higher dimensional polynomial jump diffusion such that $\epsilon_i(x) = 0$.

European call option at time $t = 0$ written on assets $S_{i,t}$ with strike price K and maturity T is given by

$$E\left[\exp(-\int_0^t r_s ds)(S_{i,t} - K)^+ \big| \mathcal{F}_0 \right] = E\left[(S_{i,0} \exp(Y_{i,T}) - K \exp(-\int_0^t r_s ds))^+ \big| \mathcal{F}_0 \right]$$

where the expectation is under the risk-neutral measure P.

Sub-fractional CKLS Model:

The CKLS model is a very popular one factor short rate model in term structure of interest rates. This general one factor interest rate model comprises of a linear drift with constant elasticity of variance. Hence this is also called the constant elasticity of variance (CEV) model or Cox-Ross model. For time varying elasticity of variance model, see Fan et al. [15]. We generalize the CKLS model to sub-fractional Brownin motion noise which will have long memory nonstationary increments. For the model to be useful and for calibration purposes, it becomes necessary to estimate the unknown parameters in the model from discrete interest rate data. See the monograph Bishwal [16] for asymptotic results on approximate likelihood estimators and approximate Bayes estimators for drift parameter estimation of discretely observed diffusions based on high frequency data. To estimate the drift parameter, we use approximate maximum likelihood estimators and study the accuracy of approximation in terms of rate of convergence of the resulting estimators from high frequency data.

On the same stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ we define the fractional CKLS process $\{X_t\}$ satisfying the fractional Itô stochastic differential equation

$$dX_t = \alpha(\beta - X_t)dt + \sigma X_t^{\gamma} dW_t^H,\ t \geq 0, \quad X_0 = x_0$$

where $\{W_t, t \geq 0\}$ is a standard sub-fractional Wiener process with the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and H is the Hurst parameter which describes the medium range memory of the process lying in $[0.5, 1]$ which is assumed to be known. The unknown parameters are α which is the mean reversion parameter, β is central tendency parameter, σ is the standard deviation or volatility and γ is the elasticity of variance. The unknown parameters are to be estimated from discrete observations of the interest rate process $\{X_t\}$. The special cases $\gamma = 0$ gives sub-fractional Vasicek model, $\gamma = 1$ gives the sub-fractional Rendleman-Bartter model and $\gamma = 1/2$ gives sub-fractional Cox-Ingersoll-Ross model which are popular one factor equilibrium models used for short term interest rate modeling, see Hull [17]. Sub-fractional Ho-Lee model and sub-fractional Hull-White models are nonhomegeneous generalizations of the sub-fractional Vasicek model, see Brigo and Mercurio [18] for the short rate dependent case $H = 0.5$. Using Itô's formula it is easy to see that the process $X_t^{1-\gamma}$ is a sub-fractional square-root diffusion process. Thus once an estimate of γ is obtained, one can simulate the process using simulation methods of square-root diffusion, see Glasserman [19].

Model parsimony is an important problem in finance. We propose the following generalization of Ait-Sahalia model (see Ait-Sahalia [20,21]) which we call the *Generalized Ait-Sahalia (GAS) Model* which is given by

$$\begin{aligned} dX_t &= (\alpha_0 + \alpha_1 X_t^{\upsilon_1} + \alpha_2 X_t^{\upsilon_2} + \alpha_3 X_t^{\upsilon_3} + \alpha_4 X_t^{\upsilon_4} (\log X_t)^{\upsilon_5} + \alpha_5 \sin X_t)\ dt \\ &\quad + \sqrt{\beta_0 + \beta_1 X_t^{\gamma_1} + \ \ \beta_2 X_t^{\gamma_2} + \beta_3 X_t^{\gamma_3} \exp(\gamma_4 X_t)}\ dW_t^H \end{aligned}$$

where W_t^H sub-fractional Brownian motion (sub-fBm).

The sine term in the model comes from exchange rate modeling by Larsen and Sørensen [13]. This is a model with 19 parameters. Special cases of this model are sub-fractional generalizations of Ait-Sahalia diffusions, Pearson diffusions, Jacobi diffusions, CIR diffusion, CKLS diffusion, CHLS diffusion (Conley et al. [27]), OU diffusion, Vasicek diffusion, Radial OU diffusion, Linetsky cube-root diffusions (see Linetsky [22]), Lewis diffusion, Ahn-Gao diffusion, Duffie-Kan diffusion, Black-Scholes diffusion, Black-Karasinski diffusion, Schwartz diffusion and Periodic diffusion.

Thus we obtain diffusions with invariant distributions as normal, log-normal, gamma, inverse gamma, non-central chi-square, skewed Student-t, scaled F, Beta, logistic, Gompertz and hyperbolic distributions. Typical value of H based on Canadian interest rate data is around 0.63.

We recall some facts from sub-fractional Pearson diffusion processes. Sub-fractional Pearson diffusion is a stationary solution to the sub-fractional stochastic differential equation of the form

$$dX_t = -\theta(X_t - \mu)dt + \sqrt{2\theta(aX_t^2 + bX_t + c)}\ dW_t^H,$$

where $\theta > 0$ and a, b and c are such that the square root is well defined when X_t is in the state space.

This is a special case of the following equation

$$dX_t = -\theta(X_t - \mu)dt + \sigma(X_t)\ dW_t^H,$$

where $\sigma^2(X_t) = 2\theta(aX_t^2 + bX_t + c)$.

For $H = 0.5$, this is a diffusion process with given invariant measure ν that admits a density p with respect to Lebesgue measure. Assume that the density p is continuous, bounded, has finite variance and p is strictly positive in the interval $(l, u)(-\infty \leq l \leq u \leq \infty)$ is zero outside (l, u). Let μ be the expectation of the measure ν and

$$\sigma^2(x) = \frac{2\int_l^x (\mu - y)p(y)dy}{p(x)} = \frac{2\mu F(x) - 2\int_l^x yp(y)dy}{p(x)},\ x \in (l, u)$$

where $F(x) = \int_{-\infty}^x p(y)dy$ is the distribution function associated with the density p.

The SDE with the above diffusion coefficient has a Markovian weak solution. We have $E(\sigma^2(X)) < \infty$ when $X \sim p$. The solution is ergodic with invariant density p.

Sub-fractional Student Diffusion The sub-fractional student diffusion is given by

$$dX_t = -\theta X_t dt + \sqrt{\frac{2\theta}{\nu - 1}(\nu + X_t^2)}\ dW_t^H$$

For $H = 0.5$, the invariant density is given by

$$f(x) = \frac{\Gamma((\nu + 1)/2)}{\sqrt{\nu\pi}\Gamma(\nu/2)}\left(1 + \frac{1}{\nu}x^2\right)^{-(\nu+1)/2},\ x \in \mathbb{R},\ \nu > 0.$$

$$= \frac{1}{\sqrt{\nu}B(\nu/2, 1/2))[1 + (x/\nu)^2]^{(\nu+1)/2}]}.$$

where $B(\cdot, \cdot)$ is the beta function.

Sub-fractional Skew Student Diffusion

$$dX_t = -\theta X_t dt + \sqrt{\frac{2\theta}{\nu - 1}\left[X_t^2 + 2\varpi\nu^{1/2}X_t + (1 + \varpi^2)\nu\right]}\ dW_t^H.$$

The parameter ϖ is the skewness parameter. The case $\varpi = 0$ gives the sub-fractional symmetric Student diffusion.

Sub-fractional Gamma Diffusion

$$dX_t = -\theta(X_t - \mu)dt + \sqrt{2\theta X_t}\ dW_t^H$$

Sub-fractional Snedecor Diffusion

$$dX_t = -\theta(X_t - \mu)dt + \sqrt{2\theta a X_t(X_t + 1)}\ dW_t^H$$

Sub-fractional Jacobi Diffusion

$$dX_t = -\theta(X_t - \mu)dt + \sqrt{2\theta a X_t(X_t - 1)}\ dW_t^H$$

Sub-fractional Uniform Diffusion

$$dX_t = -\theta(X_t - \frac{1}{2})dt + \sqrt{\theta X_t(1 - X_t)}\ dW_t^H$$

Sub-fractional Logistic Diffusion

$$dX_t = -\theta\left[1 - 2\mu + (1 - \mu)e^{X_t} - \mu e^{-1} - 8a\cosh^4\left(\frac{X_t}{2}\right)\right]dt$$

$$+2\sqrt{-a\theta}\cosh\left(\frac{X_t}{2}\right)dW_t^H$$

Sub-fractional Gompertz Diffusion

$$dX_t = (\alpha X_t - \beta X_t \ln X_t)\ dt + \sigma X_t\ dW_t^H$$

A crucial feature of the CIR model is that both the infinitesimal drift $\alpha(\beta - x)$ and the variance $\sigma^2 x$ are affine functions of the state variable. However, empirical studies contradicts the CIR specification. Using the sub-fractional Bessel process with constant drift, we find some nonaffine specifications that are analytically tractable.

Sub-fractional Non-affine Models:

Sub-fractional Bessel process:

$$dY_t = ((\nu + 1/2)Y_t^{-1} + \mu)dt + \ dW_t^H$$

Applying Itô's lemma with $X_t = 4(\sigma Y_t)^{-2}$ we obtain

$$dX_t = -\theta(X_t^{1/2} - \mu)X_t^{3/2}dt + \sigma X_t^{3/2}\ dW_t^H$$

This diffusion has nonlinear drift and infinitesimal variance $\sigma^2 x^3$.

Applying Itô's lemma with $X_t = (\sigma Y_t)^{-1}$ we obtain

$$dX_t = -\theta(X_t - \mu)X_t^2 dt + \sigma X_t^2\ dW_t^H$$

This diffusion has nonlinear drift and infinitesimal variance $\sigma^2 x^4$.

Sub-fractional Bessel process with linear drift:

$$dY_t = ((\nu + 1/2)Y_t^{-1} + \mu Y_t)dt + \ dW_t^H$$

Sub-fractional 3/2 model:

By fractional Itô's lemma, the *Sub-fractional reciprocal squared process*, i.e., taking $X_t = (4/\sigma^2)Y_t^{-2}$, we obtain the Sub-fractional 3/2 model

$$dX_t = -\theta(X_t - \mu)X_t dt + \sigma X_t^{3/2}\ dW_t^H$$

This diffusion with nonlinear drift and infinitesimal variance $\sigma^2 x^3$ is the reciprocal of the square-root CIR process. This process was proposed by Cox et al. [23] as a model for the inflation rate in their three-factor inflation rate.

Pricing Interest Rate Derivatives:

Any European style interest rate derivative with some payoff function $f(X_t)$ at anytime $t > 0$ can be valued by integrating the payoff function against the state-price density:

$$E\left[\exp\left(-\int_0^t X_u du\right) f(X_t)|X_0 = x\right] = \int_0^\infty f(y)\pi(t;x,y)dy.$$

The state-price density $\pi(t;x,y)$ can be interpreted as the transition density of the diffusion *killed at the linear rate*. The state-price density of the short rate of the above nonaffine models can be easily obtained from the transition density of the Bessel process with constant drift.

3 Newton-Cotes Distribution and Drift Estimators

Hereafter, we focus on the estimation in CKLS diffusion. The initial value of the process $x_0 \geq 0$ for nonnegativity of the solution. For simplicity of exposition, we assume $x_0 = 0$. Let the continuous realization $\{X_t, 0 \leq t \leq T\}$ be denoted by X_0^T. Let $P_{(\alpha,\beta)}^T$ be the measure generated on the space (C_T, B_T) of continuous functions on $[0,T]$ with the associated Borel σ-algebra B_T generated under the supremum norm by the process X_0^T and let P_0^T be the standard Wiener measure. It is well known that when (α,β) is the true value of the parameter vector, $P_{(\alpha,\beta)}^T$ is absolutely continuous with respect to P_0^T and the Radon-Nikodym derivative (likelihood) of $P_{(\alpha,\beta)}^T$ with respect to P_0^T based on observation X_0^T is given by

$$\begin{aligned} & L_T(\alpha,\beta) := \frac{dP_{(\alpha,\beta)}^T}{dP_0^T}(X_0^T) \\ = \quad & \exp\left\{\int_0^T \alpha(\beta - X_t)\sigma^{-2}X_t^{1-2\gamma}dX_t - \frac{1}{2}\int_0^T \alpha(\beta - X_t)^2\sigma^{-2}X_t^{-2\gamma}dt\right\}. \end{aligned}$$

Consider the score function, the derivative of the log-likelihood function, which is given by

$$l_T(\alpha,\beta) := \sigma^{-2}\left\{\int_0^T \alpha(\beta - X_t)X_t^{1-2\gamma}dX_t - \frac{1}{2}\int_0^T \alpha(\beta - X_t)^2X_t^{-2\gamma}dt\right\}.$$

A solution of the estimating equation $l_T(\alpha,\beta) = 0$ provides the conditional maximum likelihood estimates (MLEs)

$$\hat{\alpha}_T := \frac{\int_0^T X_t^{-2\gamma}dX_t \int_0^T X_t^{1-2\gamma}dt - \int_0^T X_t^{1-2\gamma}dX_t \int_0^T X_t^{-2\gamma}dt}{\int_0^T X_t^{-2\gamma}dt \int_0^T X_t^{2-2\gamma}dt - \int_0^T X_t^{1-2\gamma}dt \int_0^T X_t^{1-2\gamma}dt}$$

and

$$\hat{\beta}_T := \frac{\int_0^T X_t^{-2\gamma}dX_t \int_0^T X_t^{2-2\gamma}dt - \int_0^T X_t^{1-2\gamma}dX_t \int_0^T X_t^{1-2\gamma}dt}{\int_0^T X_t^{-2\gamma}dX_t \int_0^T X_t^{1-2\gamma}dt - \int_0^T X_t^{1-2\gamma}dX_t \int_0^T X_t^{-2\gamma}dt}.$$

We transform the Itô integrals $\int$ to the Stratonovich integrals $\oint$ as follows:

For a smooth function $f(\cdot)$, we have

$$\int_0^T f(X_t)dX_t = \oint_0^T f(X_t)dX_t - \frac{\sigma^2}{2}\int_0^T f'(X_t)X_t^{2\gamma}dt.$$

For simplicity of presentation, we assume that $\sigma = 1$. Thus

$$\int_0^T X_t^{-2\gamma}dX_t = \oint_0^T X_t^{-2\gamma}dX_t + \gamma\int_0^T X_t^{-1}dt,$$

$$\int_0^T X_t^{1-2\gamma}dX_t = \oint_0^T X_t^{1-2\gamma}dX_t - \frac{1-2\gamma}{2}T,$$

$$\int_0^T X_t^{2-2\gamma}dX_t = \oint_0^T X_t^{2-2\gamma}dX_t - (1-\gamma)\int_0^T X_t dt.$$

Thus Stratonovich integral based MLEs are given by

$$\hat{\alpha}_T := \frac{\left(\oint_0^T X_t^{-2\gamma}dX_t + \gamma\int_0^T X_t^{-1}dt\right)\int_0^T X_t^{1-2\gamma}dt - \left(\oint_0^T X_t^{1-2\gamma}dX_t - \frac{1-2\gamma}{2}T\right)\int_0^T X_t^{-2\gamma}dt}{\int_0^T X_t^{-2\gamma}dt \int_0^T X_t^{2-2\gamma}dt - \int_0^T X_t^{1-2\gamma}dt \int_0^T X_t^{1-2\gamma}dt},$$

$$\hat{\beta}_T := \frac{\left(\oint_0^T X_t^{-2\gamma} dX_t + \gamma \int_0^T X_t^{-1} dt\right) \int_0^T X_t^{2-2\gamma} dt - \left(\oint_0^T X_t^{1-2\gamma} dX_t - \frac{1-2\gamma}{2} T\right) \int_0^T X_t^{1-2\gamma} dt}{\left(\oint_0^T X_t^{-2\gamma} dX_t + \gamma \int_0^T X_t^{-1} dt\right) \int_0^T X_t^{1-2\gamma} dt - \left(\oint_0^T X_t^{1-2\gamma} dX_t - \frac{1-2\gamma}{2} T\right) \int_0^T X_t^{-2\gamma} dt}.$$

Laurini and Hotta [24] studied estimation in CIR fractional short-term interest rate models driven by fractional Brownian motion using simulation based *indirect inference method.*

In this paper, we obtain rates of convergence for several approximate maximum likelihood estimators (AMLEs) based on discrete observations of the interest rate model. We assume that the process $\{X_t\}$ is observed at discrete time points $0 = t_0 < t_1 < \cdots t_n = T$ with $t_i - t_{i-1} = \frac{T}{n}, i = 1, 2 \cdots, n$. For asymptotics, we assume two types of high frequency discrete data with long observation time: (1) $T \to \infty, n \to \infty, \frac{T}{n} \to 0$, (2) $T \to \infty, n \to \infty, \frac{T}{n^{2/3}} \to 0$. All the estimators of the drift parameters obtained in this paper are consistent under sampling scheme (1) and asymptotically normally distributed under sampling scheme (2).

If the volatility parameter σ is unknown, it can be estimated as the positive square-root of

$$\hat{\sigma}_{n,T}^2 := \frac{\sum_{i=1}^n (X_{t_i} - X_{t_{i-1}})^2}{\sum_{i=1}^n X_{t_{i-1}}^2 (t_i - t_{i-1})}.$$

For a weight function $w_{t_i} \geq 0$, define the weighted AMLEs

$$\begin{aligned}
\tilde{\alpha}_{n,T} := & \Big(\Big[\Big\{\sum_{i=1}^n w_{t_i} X_{t_{i-1}}^{-2\gamma} + \sum_{i=2}^{n+1} w_{t_i} X_{t_{i-1}}^{-2\gamma}\Big\}(X_{t_i} - X_{t_{i-1}}) \\
& + \gamma \Big\{\sum_{i=1}^n w_{t_i} X_{t_{i-1}}^{-1} + \sum_{i=2}^{n+1} w_{t_i} X_{t_{i-1}}^{-1}\Big\}(t_i - t_{i-1})\Big] \\
& \Big[\Big\{\sum_{i=1}^n w_{t_i} X_{t_{i-1}}^{1-2\gamma} + \sum_{i=2}^{n+1} w_{t_i} X_{t_{i-1}}^{1-2\gamma}\Big\}(t_i - t_{i-1}) \\
& - \Big\{\sum_{i=1}^n w_{t_i} X_{t_{i-1}}^{1-2\gamma} + \sum_{i=2}^{n+1} w_{t_i} X_{t_{i-1}}^{1-2\gamma}\Big\}(X_{t_i} - X_{t_{i-1}}) - \frac{1-2\gamma}{2} T\Big] \\
& \Big[\Big\{\sum_{i=1}^n w_{t_i} X_{t_{i-1}}^{2-2\gamma} + \sum_{i=2}^{n+1} w_{t_i} X_{t_{i-1}}^{2-2\gamma}\Big\}(t_i - t_{i-1})\Big]\Big) \\
& \Big(\Big[\Big\{\sum_{i=1}^n w_{t_i} X_{t_{i-1}}^{-2\gamma} + \sum_{i=2}^{n+1} w_{t_i} X_{t_{i-1}}^{-2\gamma}\Big\}(t_i - t_{i-1})\Big] \\
& \Big[\Big\{\sum_{i=1}^n w_{t_i} X_{t_{i-1}}^{2-2\gamma} + \sum_{i=2}^{n+1} w_{t_i} X_{t_{i-1}}^{2-2\gamma}\Big\}(t_i - t_{i-1}) \\
& - \Big\{\sum_{i=1}^n w_{t_i} X_{t_{i-1}}^{1-2\gamma} + \sum_{i=2}^{n+1} w_{t_i} X_{t_{i-1}}^{1-2\gamma}\Big\}(t_i - t_{i-1})\Big] \\
& \Big[\Big\{\sum_{i=1}^n w_{t_i} X_{t_{i-1}}^{1-2\gamma} + \sum_{i=2}^{n+1} w_{t_i} X_{t_{i-1}}^{1-2\gamma}\Big\}(t_i - t_{i-1})\Big]\Big)^{-1},
\end{aligned}$$

$$\begin{aligned}
\tilde{\beta}_{n,T} := & \Big(\Big[\Big\{\sum_{i=1}^n w_{t_i} X_{t_{i-1}}^{-2\gamma} + \sum_{i=2}^{n+1} w_{t_i} X_{t_{i-1}}^{-2\gamma}\Big\}(X_{t_i} - X_{t_{i-1}}) \\
& + \gamma \Big\{\sum_{i=1}^n w_{t_i} X_{t_{i-1}}^{-1} + \sum_{i=2}^{n+1} w_{t_i} X_{t_{i-1}}^{-1}\Big\}(t_i - t_{i-1})\Big] \\
& \Big[\Big\{\sum_{i=1}^n w_{t_i} X_{t_{i-1}}^{2-2\gamma} + \sum_{i=2}^{n+1} w_{t_i} X_{t_{i-1}}^{2-2\gamma}\Big\}(t_i - t_{i-1}) \\
& - \Big\{\sum_{i=1}^n w_{t_i} X_{t_{i-1}}^{1-2\gamma} + \sum_{i=2}^{n+1} w_{t_i} X_{t_{i-1}}^{1-2\gamma}\Big\}(X_{t_i} - X_{t_{i-1}}) - \frac{1-2\gamma}{2} T\Big] \\
& \Big[\Big\{\sum_{i=1}^n w_{t_i} X_{t_{i-1}}^{1-2\gamma} + \sum_{i=2}^{n+1} w_{t_i} X_{t_{i-1}}^{1-2\gamma}\Big\}(t_i - t_{i-1})\Big]\Big) \\
& \Big(\Big[\Big\{\sum_{i=1}^n w_{t_i} X_{t_{i-1}}^{-2\gamma} + \sum_{i=2}^{n+1} w_{t_i} X_{t_{i-1}}^{-2\gamma}\Big\}(X_{t_i} - X_{t_{i-1}})\Big] \\
& \Big[\Big\{\sum_{i=1}^n w_{t_i} X_{t_{i-1}}^{1-2\gamma} + \sum_{i=2}^{n+1} w_{t_i} X_{t_{i-1}}^{1-2\gamma}\Big\}(t_i - t_{i-1}) \\
& - \Big\{\sum_{i=1}^n w_{t_i} X_{t_{i-1}}^{1-2\gamma} + \sum_{i=2}^{n+1} w_{t_i} X_{t_{i-1}}^{1-2\gamma}\Big\}(X_{t_i} - X_{t_{i-1}})\Big] \\
& \Big[\Big\{\sum_{i=1}^n w_{t_i} X_{t_{i-1}}^{-2\gamma} + \sum_{i=2}^{n+1} w_{t_i} X_{t_{i-1}}^{-2\gamma}\Big\}(t_i - t_{i-1})\Big]\Big)^{-1}.
\end{aligned}$$

With $w_{t_i}=1$, we obtain the forward AMLEs

$$\begin{aligned}\tilde{\alpha}_{n,T,F} &:= \left[\sum_{i=1}^n X_{t_{i-1}}^{1-2\gamma}(X_{t_i}-X_{t_{i-1}}) - \tfrac{1-\gamma}{2}\sum_{i=1}^n X_{t_{i-1}}^{\gamma}(t_i-t_{i-1})\right] \\ &\quad \left[\sum_{i=1}^n X_{t_{i-1}}^{2-2\gamma}(t_i-t_{i-1})\right]^{-1}, \\ \tilde{\beta}_{n,T,F} &:= \left[\sum_{i=1}^n X_{t_{i-1}}^{1-2\gamma}(X_{t_i}-X_{t_{i-1}}) - \tfrac{1-\gamma}{2}\sum_{i=1}^n X_{t_{i-1}}^{\gamma}(t_i-t_{i-1})\right] \\ &\quad \left[\sum_{i=1}^n X_{t_{i-1}}^{2-2\gamma}(t_i-t_{i-1})\right]^{-1}.\end{aligned}$$

With $w_{t_i}=0$, we obtain the backward AMLEs

$$\begin{aligned}\tilde{\alpha}_{n,T,B} &:= \left[\sum_{i=1}^n X_{t_i}^{1-2\gamma}(X_{t_i}-X_{t_{i-1}}) - \tfrac{1-\gamma}{2}\sum_{i=1}^n X_{t_i}^{\gamma}(t_i-t_{i-1})\right] \\ &\quad \left[\sum_{i=1}^n X_{t_i}^{2-2\gamma}(t_i-t_{i-1})\right]^{-1}, \\ \tilde{\beta}_{n,T,B} &:= \left[\sum_{i=1}^n X_{t_i}^{1-2\gamma}(X_{t_i}-X_{t_{i-1}}) - \tfrac{1-\gamma}{2}\sum_{i=1}^n X_{t_i}^{\gamma}(t_i-t_{i-1})\right] \\ &\quad \left[\sum_{i=1}^n X_{t_i}^{2-2\gamma}(t_i-t_{i-1})\right]^{-1}.\end{aligned}$$

Analogous to the estimators for the discrete first order autoregressive model AR(1), we define the simple symmetric and weighted symmetric estimators as follows:

With $w_{t_i}=1/2$, the simple symmetric AMLE is defined as

$$\begin{aligned}\tilde{\alpha}_{n,T,M} &:= \left[\left\{\sum_{i=2}^n X_{t_{i-1}}^{1-2\gamma} + \tfrac{1}{2}(X_{t_0}^{1-2\gamma}+X_{t_n}^{1-2\gamma})\right\}(X_{t_i}-X_{t_{i-1}})\right. \\ &\quad \left.-\tfrac{1-\gamma}{2}\left\{\sum_{i=2}^n X_{t_{i-1}}^{\gamma} + 0.5(X_{t_0}^{\gamma}+X_{t_n}^{\gamma})\right\}(t_i-t_{i-1})\right] \\ &\quad \left[\left\{\sum_{i=2}^n X_{t_{i-1}}^{2-2\gamma} + 0.5(X_{t_0}^{2-2\gamma}+X_{t_n}^{2-2\gamma})\right\}(t_i-t_{i-1})\right]^{-1}, \\ \tilde{\beta}_{n,T,M} &:= \left[\left\{\sum_{i=2}^n X_{t_{i-1}}^{1-2\gamma} + \tfrac{1}{2}(X_{t_0}^{1-2\gamma}+X_{t_n}^{1-2\gamma})\right\}(X_{t_i}-X_{t_{i-1}})\right. \\ &\quad \left.-\tfrac{1-\gamma}{2}\left\{\sum_{i=2}^n X_{t_{i-1}}^{\gamma} + 0.5(X_{t_0}^{\gamma}+X_{t_n}^{\gamma})\right\}(t_i-t_{i-1})\right] \\ &\quad \left[\left\{\sum_{i=2}^n X_{t_{i-1}}^{2-2\gamma} + 0.5(X_{t_0}^{2-2\gamma}+X_{t_n}^{2-2\gamma})\right\}(t_i-t_{i-1})\right]^{-1}.\end{aligned}$$

With the weight function

$$w_{t_i} = \begin{cases} 0 & : \quad i=1 \\ \frac{i-1}{n} & : \quad i=2,3,\cdots,n \\ 1 & : \quad i=n+1 \end{cases}$$

the weighted symmetric AMLE is defined as

$$\begin{aligned}\tilde{\alpha}_{n,T,D} &:= \left[\left\{\sum_{i=2}^n X_{t_{i-1}}^{1-2\gamma} + \sum_{i=1}^n X_{t_{i-1}}^{1-2\gamma}\right\}(X_{t_i}-X_{t_{i-1}})\right. \\ &\quad \left.-\tfrac{1-\gamma}{2}\left\{\sum_{i=2}^n X_{t_{i-1}}^{\gamma} + \sum_{i=1}^n X_{t_{i-1}}^{\gamma}\right\}(t_i-t_{i-1})\right] \\ &\quad \left[\left\{\sum_{i=2}^n X_{t_{i-1}}^{2-2\gamma} + \sum_{i=1}^n X_{t_{i-1}}^{2-2\gamma}\right\}(t_i-t_{i-1})\right]^{-1}. \\ \tilde{\beta}_{n,T,D} &:= \left[\left\{\sum_{i=2}^n X_{t_{i-1}}^{1-2\gamma} + \sum_{i=1}^n X_{t_{i-1}}^{1-2\gamma}\right\}(X_{t_i}-X_{t_{i-1}})\right. \\ &\quad \left.-\tfrac{1-\gamma}{2}\left\{\sum_{i=2}^n X_{t_{i-1}}^{\gamma} + \sum_{i=1}^n X_{t_{i-1}}^{\gamma}\right\}(t_i-t_{i-1})\right] \\ &\quad \left[\left\{\sum_{i=2}^n X_{t_{i-1}}^{2-2\gamma} + \sum_{i=1}^n X_{t_{i-1}}^{2-2\gamma}\right\}(t_i-t_{i-1})\right]^{-1}.\end{aligned}$$

Note the above two estimators are analogous to the trapezoidal rule in numerical analysis. One can instead use the midpoint rule to define the estimators

$$\begin{aligned}\tilde{\alpha}_{n,T,A} &:= \left[\sum_{i=1}^n \left(\tfrac{X_{t_{i-1}}+X_{t_i}}{2}\right)^{1-2\gamma}(X_{t_i}-X_{t_{i-1}})\right. \\ &\quad \left.-\tfrac{1-\gamma}{2}\sum_{i=1}^n \left(\tfrac{X_{t_{i-1}}+X_{t_i}}{2}\right)^{\gamma}(t_i-t_{i-1})\right] \\ &\quad \left[\sum_{i=1}^n \left(\tfrac{X_{t_{i-1}}+X_{t_i}}{2}\right)^{2-2\gamma}(t_i-t_{i-1})\right]^{-1},\end{aligned}$$

$$\tilde{\beta}_{n,T,A} := \left[\sum_{i=1}^{n}\left(\frac{X_{t_{i-1}}+X_{t_i}}{2}\right)^{1-2\gamma}(X_{t_i}-X_{t_{i-1}}) - \frac{1-\gamma}{2}\sum_{i=1}^{n}\left(\frac{X_{t_{i-1}}+X_{t_i}}{2}\right)^{\gamma}(t_i-t_{i-1})\right]\left[\sum_{i=1}^{n}\left(\frac{X_{t_{i-1}}+X_{t_i}}{2}\right)^{2-2\gamma}(t_i-t_{i-1})\right]^{-1}.$$

One can use the Simpson's rule to define the estimators which are convex combination of trapezoidal and midpoint estimators:

$$\tilde{\alpha}_{n,T,S} := \left[\frac{1}{3}\sum_{i=1}^{n}\left\{X_{t_{i-1}}^{1-2\gamma}+4\left(\frac{X_{t_{i-1}}+X_{t_i}}{2}\right)^{1-2\gamma}+X_{t_i}^{1-2\gamma}\right\}(X_{t_i}-X_{t_{i-1}}) - \frac{1-\gamma}{6}\sum_{i=1}^{n}\left\{X_{t_{i-1}}^{\gamma}+4\left(\frac{X_{t_{i-1}}+X_{t_i}}{2}\right)^{\gamma}+X_{t_i}^{-1}\right\}(t_i-t_{i-1})\right]\left[\frac{1}{3}\sum_{i=1}^{n}\left\{X_{t_{i-1}}^{2-2\gamma}+4\left(\frac{X_{t_{i-1}}+X_{t_i}}{2}\right)^{2-2\gamma}+X_{t_i}^{2-2\gamma}\right\}(t_i-t_{i-1})\right]^{-1},$$

$$\tilde{\beta}_{n,T,S} := \left[\frac{1}{3}\sum_{i=1}^{n}\left\{X_{t_{i-1}}^{1-2\gamma}+4\left(\frac{X_{t_{i-1}}+X_{t_i}}{2}\right)^{1-2\gamma}+X_{t_i}^{1-2\gamma}\right\}(X_{t_i}-X_{t_{i-1}}) - \frac{1-\gamma}{6}\sum_{i=1}^{n}\left\{X_{t_{i-1}}^{\gamma}+4\left(\frac{X_{t_{i-1}}+X_{t_i}}{2}\right)^{\gamma}+X_{t_i}^{-1}\right\}(t_i-t_{i-1})\right]\left[\frac{1}{3}\sum_{i=1}^{n}\left\{X_{t_{i-1}}^{2-2\gamma}+4\left(\frac{X_{t_{i-1}}+X_{t_i}}{2}\right)^{2-2\gamma}+X_{t_i}^{2-2\gamma}\right\}(t_i-t_{i-1})\right]^{-1}.$$

The estimators $\tilde{\alpha}_{n,T,D}$ and $\tilde{\beta}_{n,T,D}$ are based on the arithmetic mean of the forward and the backward sum. One can use geometric mean and harmonic mean instead for positive diffusions. Note that the symmetric estimators use both the end points of observations. These estimators are sufficient statistics. So one can take advantage of Rao-Blackwell theorem. If one excludes the end points of observations (as in the forward and the backward estimators), then the estimators will not be sufficient statistics. The simple symmetric approximate MLE was introduced and studied first in Mishra and Bishwal [25] for a more general diffusion process. Stochastic bound between the difference the AMLE and the continuous MLE was obtained.

In general, one can generalize Simpson's rule for any $0 \le \theta \le 1$ as

$$\tilde{\alpha}_{n,T,GS} := \left[\sum_{i=1}^{n}\left\{\theta\frac{X_{t_{i-1}}^{1-2\gamma}+X_{t_i}^{1-2\gamma}}{2}+(1-\theta)\left(\frac{X_{t_{i-1}}+X_{t_i}}{2}\right)^{1-2\gamma}\right\}(X_{t_i}-X_{t_{i-1}}) - \frac{1-\gamma}{2}\sum_{i=1}^{n}\left\{\theta\frac{X_{t_{i-1}}^{-1}+X_{t_i}^{\gamma}}{2}+(1-\theta)\left(\frac{X_{t_{i-1}}+X_{t_i}}{2}\right)^{\gamma}\right\}(t_i-t_{i-1})\right]\left[\sum_{i=1}^{n}\left\{\theta\frac{X_{t_{i-1}}^{2-2\gamma}+X_{t_i}^{2-2\gamma}}{2}+(1-\theta)\left(\frac{X_{t_{i-1}}+X_{t_i}}{2}\right)^{2-2\gamma}\right\}(t_i-t_{i-1})\right]^{-1}$$

$$\tilde{\beta}_{n,T,GS} := \left[\sum_{i=1}^{n}\left\{\theta\frac{X_{t_{i-1}}^{1-2\gamma}+X_{t_i}^{1-2\gamma}}{2}+(1-\theta)\left(\frac{X_{t_{i-1}}+X_{t_i}}{2}\right)^{1-2\gamma}\right\}(X_{t_i}-X_{t_{i-1}}) - \frac{1-\gamma}{2}\sum_{i=1}^{n}\left\{\theta\frac{X_{t_{i-1}}^{\gamma}+X_{t_i}^{\gamma}}{2}+(1-\theta)\left(\frac{X_{t_{i-1}}+X_{t_i}}{2}\right)^{\gamma}\right\}(t_i-t_{i-1})\right]\left[\sum_{i=1}^{n}\left\{\theta\frac{X_{t_{i-1}}^{2-2\gamma}+X_{t_i}^{2-2\gamma}}{2}+(1-\theta)\left(\frac{X_{t_{i-1}}+X_{t_i}}{2}\right)^{2-2\gamma}\right\}(t_i-t_{i-1})\right]^{-1}.$$

The case $\theta = 0$ produces the A-estimators. The case $\theta = 1$ produces the M-estimators. The case $\theta = \frac{1}{3}$ produces the S-estimators.

We propose a very general form of the quadrature based estimator as

$$\begin{aligned}\tilde{\beta}_{n,T,Q} &:= \Big[\sum_{i=1}^{n}\sum_{j=1}^{m}\left[(1-s_j)X_{t_{i-1}}+s_jX_{t_i}\right]^{1-2\gamma}p_j\,(X_{t_i}-X_{t_{i-1}}) \\ &\quad -\tfrac{1-\gamma}{2}\left\{\sum_{i=1}^{n}\sum_{j=1}^{m}\left[(1-s_j)X_{t_{i-1}}+s_jX_{t_i}\right]^{\gamma}p_j\right\}(t_i-t_{i-1})\Big] \\ &\quad \Big[\left\{\sum_{i=1}^{n}\sum_{j=1}^{m}\left[(1-s_j)X_{t_{i-1}}+s_jX_{t_i}\right]^{1-2\gamma}p_j\right\}(t_i-t_{i-1})\Big]^{-1}\end{aligned}$$

where $p_j,\ \ j\in\{1,2,\cdots,m\}$ is a probability mass function of a discrete random variable S on $0\le s_1<s_2<\cdots<s_m\le 1$ with $P(S=s_j):=p_j,\ j\in\{1,2,\cdots,m\}$. Denote the k-th moment of the random variable S as $\mu_k:=\sum_{j=1}^{m}s_j^kp_j,\ k=1,2,\cdots$.

If one chooses the probability distribution as uniform distribution for which the moments are a harmonic sequence $(\mu_1,\mu_2,\mu_3,\mu_4,\mu_5,\mu_6,\cdots)=(\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5},\frac{1}{6},\frac{1}{7},\cdots)$ then there is no change in rate of convergence than second order. If one can construct a probability distribution for which the harmonic sequence is truncated at a point, then there is an improvement in the rate of convergence at the point of truncation. We conjecture the following construction:

Given a positive integer m, construct a probability mass function $p_j,\ \ j\in\{1,2,\cdots,m\}$ on $0\le s_1<s_2<\cdots<s_m\le 1$ such that

$$\sum_{j=1}^{m}s_j^rp_j=\frac{1}{r+1},\ r\in\{0,\cdots,m-2\},\quad \sum_{j=1}^{m}s_j^{m-1}p_j\ne\frac{1}{m}.$$

Neither the probabilities p_j nor the atoms, s_j, of the distribution are specified in advance.

This problem is related to the truncated Hausdorff moment problem. We obtain examples of such probability distributions and use them to get higher order accurate (up to sixth order) AMLEs.

The order of approximation error (rate of convergence) of an estimator is $n^{-\nu}$ where

$$\nu:=\inf\left\{k:\mu_k\ne\frac{1}{1+k},\ \mu_j=\frac{1}{1+j},j=1,2,\cdots,k-1\right\}.$$

In order to obtain higher order approximations, one needs to concentrate on higher order moments of the weighting probability distribution.

Newton-Cotes Probability Distribution

Let δ_a be a Dirac measure at the point a. For $N\ge 2$, the Newton-Cotes distribution is given by

$$\nu_N=\sum_{j=0}^{2N-2}\gamma_j\delta_{j/(2N-2)}\ \text{ where }\ \gamma_j=\int_0^1\prod_{k\ne j}\frac{2(N-1)u-k}{j-k}du.$$

For $N=0$, $\nu_0=\delta_0$. For $N=1$, $\nu_1=(\delta_0+\delta_1)/2$, the trapezoidal distribution. For $N=2$, the Simpson's distribution is given by

$$\nu_2=\sum_{j=0}^{2}\gamma_j\delta_{j/2}\ \text{ where }\ \gamma_j=\int_0^1\prod_{k\ne j}\frac{2u-k}{j-k}du.$$

Simply

$$\nu_2=\gamma_0\delta_0+\gamma_1\delta_{1/2}+\gamma_2\delta_1\ \text{ where }\ \gamma_0=\frac{1}{6},\ \gamma_1=\frac{2}{6},\ \gamma_2=\frac{1}{6}.$$

Boole's Probability Distribution

The Boole distribution is defined as

$$\nu_3 = \sum_{j=0}^{4} \gamma_j \delta_{j/4} \quad \text{where} \quad \gamma_j = \int_0^1 \prod_{k \neq j} \frac{4u-k}{j-k} du.$$

Simply

$$\nu_3 = \gamma_0\delta_0 + \gamma_1\delta_{1/4} + \gamma_2\delta_{1/2} + \gamma_3\delta_{3/4} + \gamma_4\delta_1$$

where

$$\gamma_0 = \frac{7}{90}, \gamma_1 = \frac{32}{90}, \gamma_2 = \frac{12}{90}, \gamma_3 = \frac{32}{90}, \gamma_4 = \frac{7}{90}.$$

The Boole probability distribution is a symmetric probability distribution with support: $(0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1)$, probability mass: $(\frac{7}{90}, \frac{32}{90}, \frac{12}{90}, \frac{32}{90}, \frac{7}{90})$ and moments: $\mu_1 = \frac{1}{2}, \mu_2 = \frac{1}{3}, \mu_3 = \frac{1}{4}, \mu_4 = \frac{1}{5}, \mu_5 = \frac{1}{6}, \mu_6 = \frac{110}{768}$. Here $\mu_6 \neq \frac{1}{7}$. Thus rate is $\nu = 6$.

Define the Boole's approximation of the stochastic integral $B_T := \oint_0^T f(X_t)dW_t$ as

$$\begin{aligned} B_{n,T} \quad := \quad & \frac{1}{90}\sum_{i=1}^{n} \left[7f(t_{i-1}, X_{t_{i-1}}) + 32f\left(\frac{3t_{i-1}+t_i}{4}, \frac{3X_{t_{i-1}}+X_{t_i}}{4}\right)\right. \\ & + 12f\left(\frac{t_{i-1}+t_i}{2}, \frac{X_{t_{i-1}}+X_{t_i}}{2}\right) + 32f\left(\frac{t_{i-1}+3t_i}{4}, \frac{X_{t_{i-1}}+3X_{t_i}}{4}\right) \\ & \left. + 7f(t_i, X_{t_i})\right](W_{t_i} - W_{t_{i-1}}). \end{aligned}$$

For fixed T,

$$B_{n,T} \xrightarrow{L_2} B_T \text{ as } n \to \infty.$$

The L_2-rate of convergence of this discrete approximation $E|B_{n,T} - B_T|^2$ is $O(n^{-6})$.

We construct probability distributions satisfying these moment conditions and obtain estimators of the rate of convergence up to order 6. Probability $p_1 = 1$ at the point $s_1 = 0$ produces the F-estimators for which $\mu_1 = 0$. Note that $\mu_1 \neq \frac{1}{2}$. Thus $\nu = 1$. Probability $p_1 = 1$ at the point $s_1 = 1$ produces the B-estimators for which $\mu_1 = 1$. Note that $\mu_1 \neq \frac{1}{2}$. Thus $\nu = 1$. Probabilities $(p_1, p_2) = (\frac{1}{2}, \frac{1}{2})$ at the respective points $(s_1, s_2) = (0, 1)$ produce the estimators $\tilde{\alpha}_{n,T,D}$ and $\tilde{\beta}_{n,T,D}$ for which $(\mu_1, \mu_2) = (\frac{1}{2}, \frac{1}{4})$. Thus $\nu = 2$. Probability $p_j = 1$ at the point $s_j = \frac{1}{2}$ produces the estimators $\tilde{\alpha}_{n,T,A}$ and $\tilde{\beta}_{n,T,A}$ for which $(\mu_1, \mu_2) = (\frac{1}{2}, \frac{1}{2})$. Thus $\nu = 2$. Probabilities $(p_1, p_2) = (\frac{1}{4}, \frac{3}{4})$ at the respective points $(s_1, s_2) = (0, \frac{2}{3})$ produce the asymmetric estimators $\tilde{\alpha}_{n,T,3}$ and $\tilde{\beta}_{n,T,3}$ for which $(\mu_1, \mu_2, \mu_3) = (\frac{1}{2}, \frac{1}{3}, \frac{2}{9})$. Thus $\nu = 3$. Probabilities $(p_1, p_2) = (\frac{3}{4}, \frac{1}{4})$ at the respective points $(s_1, s_2) = (\frac{1}{3}, 1)$ produce the asymmetric estimators $\tilde{\alpha}_{n,T,4}$ and $\tilde{\beta}_{n,T,4}$ for which $(\mu_1, \mu_2, \mu_3) = (\frac{1}{2}, \frac{1}{3}, \frac{10}{36})$. Thus $\nu = 3$. Probabilities $(p_1, p_2, p_3) = (\frac{1}{6}, \frac{2}{3}, \frac{1}{6})$ at the respective points $(s_1, s_2, s_3) = (0, \frac{1}{2}, 1)$ produce the estimators $\tilde{\alpha}_{n,T,5}$ and $\tilde{\beta}_{n,T,5}$ for which $(\mu_1, \mu_2, \mu_3, \mu_4) = (\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{5}{25})$. Thus $\nu = 4$. Probabilities $(p_1, p_2, p_3, p_4) = (\frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8})$ at the respective points $(s_1, s_2, s_3, s_4) = (0, \frac{1}{3}, \frac{2}{3}, 1)$ produce the symmetric estimators $\tilde{\alpha}_{n,T,6}$ and $\tilde{\beta}_{n,T,6}$ for which $(\mu_1, \mu_2, \mu_3, \mu_4) = (\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{11}{54})$. Thus $\nu = 4$. Probabilities $(p_1, p_2, p_3, p_4, p_5) = (\frac{1471}{24192}, \frac{6925}{24192}, \frac{1475}{12096}, \frac{2725}{12096}, \frac{5675}{24192}, \frac{1721}{24192})$ at the respective points $(s_1, s_2, s_3, s_4, s_5) = (0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1)$ produce the asymmetric estimators $\tilde{\alpha}_{n,T,7}$ and $\tilde{\beta}_{n,T,7}$ for which $(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6) = (\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{841}{5040})$. Thus $\nu = 5$. Probabilities $(p_1, p_2, p_3, p_4, p_5) = (\frac{7}{90}, \frac{16}{45}, \frac{2}{15}, \frac{16}{45}, \frac{7}{90})$ at the respective points $(s_1, s_2, s_3, s_4, s_5) = (0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1)$ produce the symmetric estimators $\tilde{\alpha}_{n,T,8}$ and $\tilde{\beta}_{n,T,8}$ for which $(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6) = (\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{110}{768})$. Thus $\nu = 6$. Probabilities $(p_1, p_2, p_3, p_4, p_5) = (\frac{19}{288}, \frac{75}{288}, \frac{50}{288}, \frac{50}{288}, \frac{75}{288}, \frac{19}{288})$ at the respective points $(s_1, s_2, s_3, s_4, s_5) = (0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1)$ produce symmetric estimators $\tilde{\alpha}_{n,T,9}$ and $\tilde{\beta}_{n,T,9}$ for which $(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6) = (\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{3219}{22500})$. Thus $\nu = 6$.

Now we are ready to present the estimators.

$$\tilde{\alpha}_{n,T,3} := \Bigg(\Bigg[\Bigg\{ \frac{1}{4} \sum_{i=1}^{n} \Big[(X_{t_{i-1}})^{1-2\gamma} + 3(\frac{X_{t_{i-1}} + 2X_{t_i}}{3})^{1-2\gamma} \Big] \Bigg\} (X_{t_i} - X_{t_{i-1}}) + \gamma \Bigg\{ \frac{1}{4} \sum_{i=1}^{n} \Big[(X_{t_{i-1}})^{-1} + 3(\frac{X_{t_{i-1}} + 2X_{t_i}}{3})^{-1} \Big] \Bigg\} (t_i - t_{i-1}) \Bigg] \Bigg[\Bigg\{ \frac{1}{4} \sum_{i=1}^{n} \Big[(X_{t_{i-1}})^{1-2\gamma} + 3(\frac{X_{t_{i-1}} + 2X_{t_i}}{3})^{1-2\gamma} \Big] \Bigg\} (X_{t_i} - X_{t_{i-1}}) \Bigg] - \Bigg\{ \frac{1}{4} \sum_{i=1}^{n} \Big[(X_{t_{i-1}})^{1-2\gamma} + 3(\frac{X_{t_{i-1}} + 2X_{t_i}}{3})^{1-2\gamma} \Big] \Bigg\} (X_{t_i} - X_{t_{i-1}}) - \frac{1-2\gamma}{2} T \Bigg] \Bigg[\Bigg\{ \frac{1}{4} \sum_{i=1}^{n} \Big[(X_{t_{i-1}})^{-2\gamma} + 3(\frac{X_{t_{i-1}} + 2X_{t_i}}{3})^{-2\gamma} \Big] \Bigg\} (t_i - t_{i-1}) \Bigg] \Bigg) \Bigg(\Bigg[\Bigg\{ \frac{1}{4} \sum_{i=1}^{n} \Big[(X_{t_{i-1}})^{-2\gamma} + 3(\frac{X_{t_{i-1}} + 2X_{t_i}}{3})^{-2\gamma} \Big] \Bigg\} (t_i - t_{i-1}) \Bigg] \Bigg[\Bigg\{ \frac{1}{4} \sum_{i=1}^{n} \Big[(X_{t_{i-1}})^{2-2\gamma} + 3(\frac{X_{t_{i-1}} + 2X_{t_i}}{3})^{2-2\gamma} \Big] \Bigg\} (t_i - t_{i-1}) \Bigg] - \Bigg[\Bigg\{ \frac{1}{4} \sum_{i=1}^{n} \Big[(X_{t_{i-1}})^{1-2\gamma} + 3(\frac{X_{t_{i-1}} + 2X_{t_i}}{3})^{1-2\gamma} \Big] \Bigg\} (t_i - t_{i-1}) \Bigg]^2 \Bigg)^{-1},$$

$$\tilde{\beta}_{n,T,3} := \Bigg(\Bigg[\Bigg\{ \frac{1}{4} \sum_{i=1}^{n} \Big[(X_{t_{i-1}})^{-2\gamma} + 3(\frac{X_{t_{i-1}} + 2X_{t_i}}{3})^{-2\gamma} \Big] \Bigg\} (X_{t_i} - X_{t_{i-1}}) + \gamma \Bigg\{ \frac{1}{4} \sum_{i=1}^{n} \Big[(X_{t_{i-1}})^{-1} + 3(\frac{X_{t_{i-1}} + 2X_{t_i}}{3})^{-1} \Big] \Bigg\} (t_i - t_{i-1}) \Bigg] \Bigg[\Bigg\{ \frac{1}{4} \sum_{i=1}^{n} \Big[(X_{t_{i-1}})^{2-2\gamma} + 3(\frac{X_{t_{i-1}} + 2X_{t_i}}{3})^{2-2\gamma} \Big] \Bigg\} (t_i - t_{i-1}) \Bigg] - \Bigg\{ \frac{1}{4} \sum_{i=1}^{n} \Big[(X_{t_{i-1}})^{1-2\gamma} + 3(\frac{X_{t_{i-1}} + 2X_{t_i}}{3})^{1-2\gamma} \Big] \Bigg\} (X_{t_i} - X_{t_{i-1}}) - \frac{1-2\gamma}{2} T \Bigg] \Bigg[\Bigg\{ \frac{1}{4} \sum_{i=1}^{n} \Big[(X_{t_{i-1}})^{-2\gamma} + 3(\frac{X_{t_{i-1}} + 2X_{t_i}}{3})^{-2\gamma} \Big] \Bigg\} (t_i - t_{i-1}) \Bigg] \Bigg) \Bigg(\Bigg[\Bigg\{ \frac{1}{4} \sum_{i=1}^{n} \Big[(X_{t_{i-1}})^{-2\gamma} + 3(\frac{X_{t_{i-1}} + 2X_{t_i}}{3})^{-2\gamma} \Big] \Bigg\} (X_{t_i} - X_{t_{i-1}}) \Bigg] + \gamma \Bigg\{ \frac{1}{4} \sum_{i=1}^{n} \Big[(X_{t_{i-1}})^{-1} + 3(\frac{X_{t_{i-1}} + 2X_{t_i}}{3})^{-1} \Big] \Bigg\} (t_i - t_{i-1}) \Bigg] \Bigg[\Bigg\{ \frac{1}{4} \sum_{i=1}^{n} \Big[(X_{t_{i-1}})^{1-2\gamma} + 3(\frac{X_{t_{i-1}} + 2X_{t_i}}{3})^{1-2\gamma} \Big] \Bigg\} (t_i - t_{i-1}) \Bigg] - \Bigg(\Bigg\{ \frac{1}{4} \sum_{i=1}^{n} \Big[(X_{t_{i-1}})^{1-2\gamma} + 3(\frac{X_{t_{i-1}} + 2X_{t_i}}{3})^{1-2\gamma} \Big] \Bigg\} (X_{t_i} - X_{t_{i-1}}) - \frac{1-2\gamma}{2} T \Bigg]$$

$$\left[\left\{\frac{1}{4}\sum_{i=1}^{n}\left[(X_{t_{i-1}})^{-2\gamma}+3(\frac{X_{t_{i-1}}+2X_{t_i}}{3})^{-2\gamma}\right]\right\}(t_i-t_{i-1})\right]\Bigg)^{-1}$$

$$\begin{aligned}
\tilde{\alpha}_{n,T,4} := &\Bigg(\Bigg[\left\{\frac{1}{4}\sum_{i=1}^{n}\left[3(\frac{2X_{t_{i-1}}+X_{t_i}}{3})^{-2\gamma}+(X_{t_i})^{-2\gamma}\right]\right\}(X_{t_i}-X_{t_{i-1}})\\
&+\gamma\left\{\frac{1}{4}\sum_{i=1}^{n}\left[3(\frac{2X_{t_{i-1}}+X_{t_i}}{3})^{-1}+(X_{t_i})^{-1}\right]\right\}(t_i-t_{i-1})\Bigg]\\
&\Bigg[\left\{\frac{1}{4}\sum_{i=1}^{n}\left[3(\frac{2X_{t_{i-1}}+X_{t_i}}{3})^{1-2\gamma}+(X_{t_i})^{1-2\gamma}\right]\right\}(t_i-t_{i-1})\Bigg]\\
&-\Bigg[\left\{\frac{1}{4}\sum_{i=1}^{n}\left[3(\frac{2X_{t_{i-1}}+X_{t_i}}{3})^{1-2\gamma}+(X_{t_i})^{1-2\gamma}\right]\right\}(X_{t_i}-X_{t_{i-1}})-\frac{1-2\gamma}{2}T\Bigg]\\
&\Bigg[\left\{\frac{1}{4}\sum_{i=1}^{n}\left[3(\frac{2X_{t_{i-1}}+X_{t_i}}{3})^{-2\gamma}+(X_{t_i})^{-2\gamma}\right]\right\}(t_i-t_{i-1})\Bigg]\Bigg)\\
&\Bigg(\Bigg[\left\{\frac{1}{4}\sum_{i=1}^{n}\left[3(\frac{2X_{t_{i-1}}+X_{t_i}}{3})^{-2\gamma}+(X_{t_i})^{-2\gamma}\right]\right\}(t_i-t_{i-1})\Bigg]\\
&\Bigg[\left\{\frac{1}{4}\sum_{i=1}^{n}\left[3(\frac{2X_{t_{i-1}}+X_{t_i}}{3})^{2-2\gamma}+(X_{t_i})^{2-2\gamma}\right]\right\}(t_i-t_{i-1})\Bigg]\\
&-\Bigg[\left\{\frac{1}{4}\sum_{i=1}^{n}\left[3(\frac{2X_{t_{i-1}}+X_{t_i}}{3})^{1-2\gamma}+(X_{t_i})^{1-2\gamma}\right]\right\}(t_i-t_{i-1})\Bigg]^2\Bigg)^{-1},
\end{aligned}$$

$$\begin{aligned}
\tilde{\beta}_{n,T,4} := &\Bigg(\Bigg[\left\{\frac{1}{4}\sum_{i=1}^{n}\left[3(\frac{2X_{t_{i-1}}+X_{t_i}}{3})^{-2\gamma}+(X_{t_i})^{-2\gamma}\right]\right\}(X_{t_i}-X_{t_{i-1}})\\
&+\gamma\left\{\frac{1}{4}\sum_{i=1}^{n}\left[3(\frac{2X_{t_{i-1}}+X_{t_i}}{3})^{-1}+(X_{t_i})^{-1}\right]\right\}(t_i-t_{i-1})\Bigg]\\
&\Bigg[\left\{\frac{1}{4}\sum_{i=1}^{n}\left[3(\frac{2X_{t_{i-1}}+X_{t_i}}{3})^{2-2\gamma}+(X_{t_i})^{2-2\gamma}\right]\right\}(t_i-t_{i-1})\Bigg]\\
&-\Bigg[\left\{\frac{1}{4}\sum_{i=1}^{n}\left[3(\frac{2X_{t_{i-1}}+X_{t_i}}{3})^{1-2\gamma}+(X_{t_i})^{1-2\gamma}\right]\right\}(X_{t_i}-X_{t_{i-1}})-\frac{1-2\gamma}{2}T\Bigg]\\
&\Bigg[\left\{\frac{1}{4}\sum_{i=1}^{n}\left[3(\frac{2X_{t_{i-1}}+X_{t_i}}{3})^{1-2\gamma}+(X_{t_i})^{1-2\gamma}\right]\right\}(t_i-t_{i-1})\Bigg]\Bigg)\\
&\Bigg(\Bigg[\left\{\frac{1}{4}\sum_{i=1}^{n}\left[3(\frac{2X_{t_{i-1}}+X_{t_i}}{3})^{-2\gamma}+(X_{t_i})^{-2\gamma}\right]\right\}(t_i-t_{i-1})\Bigg]\\
&+\gamma\Bigg[\left\{\frac{1}{4}\sum_{i=1}^{n}\left[3(\frac{2X_{t_{i-1}}+X_{t_i}}{3})^{-1}+(X_{t_i})^{-1}\right]\right\}(t_i-t_{i-1})\Bigg]\\
&\Bigg[\left\{\frac{1}{4}\sum_{i=1}^{n}\left[3(\frac{2X_{t_{i-1}}+X_{t_i}}{3})^{1-2\gamma}+(X_{t_i})^{1-2\gamma}\right]\right\}(t_i-t_{i-1})\Bigg]\\
&-\Bigg[\left\{\frac{1}{4}\sum_{i=1}^{n}\left[3(\frac{2X_{t_{i-1}}+X_{t_i}}{3})^{1-2\gamma}+(X_{t_i})^{1-2\gamma}\right]\right\}(t_i-t_{i-1})-\frac{1-2\gamma}{2}T\Bigg]
\end{aligned}$$

$$\left[\left\{\frac{1}{4}\sum_{i=1}^{n}\left[3(\frac{2X_{t_{i-1}}+X_{t_i}}{3})^{-2\gamma}+(X_{t_i})^{-2\gamma}\right]\right\}(t_i-t_{i-1})\right]^2\Bigg)^{-1}$$

$\tilde{\alpha}_{n,T,6} :=$

$$\Bigg(\left[\left\{\frac{1}{8}\sum_{i=1}^{n}\left[(X_{t_{i-1}})^{-2\gamma}+3(\frac{2X_{t_{i-1}}+X_{t_i}}{3})^{-2\gamma}+3(\frac{X_{t_{i-1}}+2X_{t_i}}{3})^{-2\gamma}+(X_{t_i})^{-2\gamma}\right]\right\}(X_{t_i}-X_{t_{i-1}})$$
$$+\gamma\left\{\frac{1}{8}\sum_{i=1}^{n}\left[(X_{t_{i-1}})^{-1}+3(\frac{2X_{t_{i-1}}+X_{t_i}}{3})^{-1}+3(\frac{X_{t_{i-1}}+2X_{t_i}}{3})^{-1}+(X_{t_i})^{-1}\right]\right\}(t_i-t_{i-1})\Bigg]$$
$$\left[\left\{\frac{1}{8}\sum_{i=1}^{n}\left[(X_{t_{i-1}})^{2-2\gamma}+3(\frac{2X_{t_{i-1}}+X_{t_i}}{3})^{2-2\gamma}+3(\frac{X_{t_{i-1}}+2X_{t_i}}{3})^{2-2\gamma}+(X_{t_i})^{2-2\gamma}\right]\right\}(t_i-t_{i-1})\right]$$
$$-\left[\frac{1}{8}\sum_{i=1}^{n}\left[(X_{t_{i-1}})^{1-2\gamma}+3(\frac{2X_{t_{i-1}}+X_{t_i}}{3})^{1-2\gamma}+3(\frac{X_{t_{i-1}}+2X_{t_i}}{3})^{1-2\gamma}+(X_{t_i})^{1-2\gamma}\right](X_{t_i}-X_{t_{i-1}})-\frac{1-2\gamma}{2}T\right]$$
$$\left[\left\{\frac{1}{8}\sum_{i=1}^{n}\left[(X_{t_{i-1}})^{1-2\gamma}+3(\frac{2X_{t_{i-1}}+X_{t_i}}{3})^{1-2\gamma}+3(\frac{X_{t_{i-1}}+2X_{t_i}}{3})^{1-2\gamma}+(X_{t_i})^{1-2\gamma}\right]\right\}(t_i-t_{i-1})\right]\Bigg)$$
$$\Bigg(\left[\left\{\frac{1}{8}\sum_{i=1}^{n}\left[(X_{t_{i-1}})^{-2\gamma}+3(\frac{2X_{t_{i-1}}+X_{t_i}}{3})^{-2\gamma}+3(\frac{X_{t_{i-1}}+2X_{t_i}}{3})^{-2\gamma}+(X_{t_i})^{-2\gamma}\right]\right\}(X_{t_i}-X_{t_{i-1}})$$
$$+\gamma\left\{\frac{1}{8}\sum_{i=1}^{n}\left[(X_{t_{i-1}})^{-1}+3(\frac{2X_{t_{i-1}}+X_{t_i}}{3})^{-1}+3(\frac{X_{t_{i-1}}+2X_{t_i}}{3})^{-1}+(X_{t_i})^{-1}\right]\right\}(t_i-t_{i-1})\Bigg]$$
$$\left[\left\{\frac{1}{8}\sum_{i=1}^{n}\left[(X_{t_{i-1}})^{1-2\gamma}+3(\frac{2X_{t_{i-1}}+X_{t_i}}{3})^{1-2\gamma}+3(\frac{X_{t_{i-1}}+2X_{t_i}}{3})^{1-2\gamma}+(X_{t_i})^{1-2\gamma}\right]\right\}(t_i-t_{i-1})\right]$$
$$-\left[\frac{1}{8}\sum_{i=1}^{n}\left[(X_{t_{i-1}})^{1-2\gamma}+3(\frac{2X_{t_{i-1}}+X_{t_i}}{3})^{1-2\gamma}+3(\frac{X_{t_{i-1}}+2X_{t_i}}{3})^{1-2\gamma}+(X_{t_i})^{1-2\gamma}\right](t_i-t_{i-1})\right]^2\Bigg)^{-1},$$

$\tilde{\beta}_{n,T,6} :=$

$$\Bigg(\left[\left\{\frac{1}{8}\sum_{i=1}^{n}\left[(X_{t_{i-1}})^{-2\gamma}+3(\frac{2X_{t_{i-1}}+X_{t_i}}{3})^{-2\gamma}+3(\frac{X_{t_{i-1}}+2X_{t_i}}{3})^{-2\gamma}+(X_{t_i})^{-2\gamma}\right]\right\}(X_{t_i}-X_{t_{i-1}})$$
$$+\gamma\left\{\frac{1}{8}\sum_{i=1}^{n}\left[(X_{t_{i-1}})^{-1}+3(\frac{2X_{t_{i-1}}+X_{t_i}}{3})^{-1}+3(\frac{X_{t_{i-1}}+2X_{t_i}}{3})^{-1}+(X_{t_i})^{-1}\right]\right\}(t_i-t_{i-1})\Bigg]$$
$$\left[\left\{\frac{1}{8}\sum_{i=1}^{n}\left[(X_{t_{i-1}})^{1-2\gamma}+3(\frac{2X_{t_{i-1}}+X_{t_i}}{3})^{1-2\gamma}+3(\frac{X_{t_{i-1}}+2X_{t_i}}{3})^{1-2\gamma}+(X_{t_i})^{1-2\gamma}\right]\right\}(t_i-t_{i-1})\right]$$
$$-\left[\frac{1}{8}\sum_{i=1}^{n}\left[(X_{t_{i-1}})^{1-2\gamma}+3(\frac{2X_{t_{i-1}}+X_{t_i}}{3})^{1-2\gamma}+3(\frac{X_{t_{i-1}}+2X_{t_i}}{3})^{1-2\gamma}+(X_{t_i})^{1-2\gamma}\right](X_{t_i}-X_{t_{i-1}})-\frac{1-2\gamma}{2}T\right]$$
$$\left[\left\{\frac{1}{8}\sum_{i=1}^{n}\left[(X_{t_{i-1}})^{-2\gamma}+3(\frac{2X_{t_{i-1}}+X_{t_i}}{3})^{-2\gamma}+3(\frac{X_{t_{i-1}}+2X_{t_i}}{3})^{-2\gamma}+(X_{t_i})^{-2\gamma}\right]\right\}(t_i-t_{i-1})\right]\Bigg)$$
$$\Bigg(\left[\left\{\frac{1}{8}\sum_{i=1}^{n}\left[(X_{t_{i-1}})^{-2\gamma}+3(\frac{2X_{t_{i-1}}+X_{t_i}}{3})^{-2\gamma}+3(\frac{X_{t_{i-1}}+2X_{t_i}}{3})^{-2\gamma}+(X_{t_i})^{-2\gamma}\right]\right\}(X_{t_i}-X_{t_{i-1}})$$
$$+\gamma\left\{\frac{1}{8}\sum_{i=1}^{n}\left[(X_{t_{i-1}})^{-1}+3(\frac{2X_{t_{i-1}}+X_{t_i}}{3})^{-1}+3(\frac{X_{t_{i-1}}+2X_{t_i}}{3})^{-1}+(X_{t_i})^{-1}\right]\right\}(t_i-t_{i-1})\Bigg]$$
$$\left[\left\{\frac{1}{8}\sum_{i=1}^{n}\left[(X_{t_{i-1}})^{1-2\gamma}+3(\frac{2X_{t_{i-1}}+X_{t_i}}{3})^{1-2\gamma}+3(\frac{X_{t_{i-1}}+2X_{t_i}}{3})^{1-2\gamma}+(X_{t_i})^{1-2\gamma}\right]\right\}(t_i-t_{i-1})\right]$$
$$-\left[\frac{1}{8}\sum_{i=1}^{n}\left[(X_{t_{i-1}})^{1-2\gamma}+3(\frac{2X_{t_{i-1}}+X_{t_i}}{3})^{1-2\gamma}+3(\frac{X_{t_{i-1}}+2X_{t_i}}{3})^{1-2\gamma}+(X_{t_i})^{1-2\gamma}\right](X_{t_i}-X_{t_{i-1}})-\frac{1-2\gamma}{2}T\right]$$
$$\left[\left\{\frac{1}{8}\sum_{i=1}^{n}\left[(X_{t_{i-1}})^{-2\gamma}+3(\frac{2X_{t_{i-1}}+X_{t_i}}{3})^{-2\gamma}+3(\frac{X_{t_{i-1}}+2X_{t_i}}{3})^{-2\gamma}+(X_{t_i})^{-2\gamma}\right]\right\}(t_i-t_{i-1})\right]\Bigg)^{-1}$$

$$\tilde{\alpha}_{n,T,7} := \Bigg(\Bigg[\Bigg\{\frac{1}{24192}\sum_{i=1}^{n}\Bigg[1471(X_{t_{i-1}})^{-2\gamma}+6925(\frac{X_{t_{i-1}}+X_{t_i}}{5})^{-2\gamma}+2950(\frac{2X_{t_{i-1}}+2X_{t_i}}{5})^{-2\gamma}$$

$$+5450(\frac{3X_{t_{i-1}}+3X_{t_i}}{5})-2\gamma+5675(\frac{4X_{t_{i-1}}+4X_{t_i}}{5})^{-2\gamma}+1721(X_{t_i})^{-2\gamma}\Big]\Big\}(X_{t_i}-X_{t_{i-1}})$$
$$+\gamma\Big\{\frac{1}{24192}\sum_{i=1}^{n}\Big[1471(X_{t_{i-1}})^{-1}+6925(\frac{X_{t_{i-1}}+X_{t_i}}{5})^{-1}+2950(\frac{2X_{t_{i-1}}+2X_{t_i}}{5})^{-1}$$
$$+5450(\frac{3X_{t_{i-1}}+3X_{t_i}}{5})^{-1}+5675(\frac{4X_{t_{i-1}}+4X_{t_i}}{5})^{-1}+1721(X_{t_i})^{-1}\Big]\Big\}(t_i-t_{i-1})\Big]$$
$$-\Big[\Big\{\frac{1}{24192}\sum_{i=1}^{n}\Big[1471(X_{t_{i-1}})^{1-2\gamma}+6925(\frac{X_{t_{i-1}}+X_{t_i}}{5})^{1-2\gamma}+2950(\frac{2X_{t_{i-1}}+2X_{t_i}}{5})^{1-2\gamma}$$
$$+5450(\frac{3X_{t_{i-1}}+3X_{t_i}}{5})^{1-2\gamma}+5675(\frac{4X_{t_{i-1}}+4X_{t_i}}{5})^{1-2\gamma}+1721(X_{t_i})^{2}\Big]\Big\}(X_{t_i}-X_{t_{i-1}})-\frac{1-2\gamma}{2}T\Big]$$
$$\Big[\Big\{\frac{1}{24192}\sum_{i=1}^{n}\Big[1471(X_{t_{i-1}})^{-2\gamma}+6925(\frac{X_{t_{i-1}}+X_{t_i}}{5})^{-2\gamma}+2950(\frac{2X_{t_{i-1}}+2X_{t_i}}{5})^{-2\gamma}$$
$$+5450(\frac{3X_{t_{i-1}}+3X_{t_i}}{5})^{-2\gamma}+5675(\frac{4X_{t_{i-1}}+4X_{t_i}}{5})^{-2\gamma}+1721(X_{t_i})^{2}\Big]\Big\}(t_i-t_{i-1})\Big]\Big)$$
$$\Big(\Big[\Big\{\frac{1}{24192}\sum_{i=1}^{n}\Big[1471(X_{t_{i-1}})^{-2\gamma}+6925(\frac{X_{t_{i-1}}+X_{t_i}}{5})^{-2\gamma}+2950(\frac{2X_{t_{i-1}}+2X_{t_i}}{5})^{-2\gamma}$$
$$+5450(\frac{3X_{t_{i-1}}+3X_{t_i}}{5})^{2}+5675(\frac{4X_{t_{i-1}}+4X_{t_i}}{5})^{-2\gamma}+1721(X_{t_i})^{-2\gamma}\Big]\Big\}(t_i-t_{i-1})\Big]$$
$$\Big[\Big\{\frac{1}{24192}\sum_{i=1}^{n}\Big[1471(X_{t_{i-1}})^{2-2\gamma}+6925(\frac{X_{t_{i-1}}+X_{t_i}}{5})^{2-2\gamma}+2950(\frac{2X_{t_{i-1}}+2X_{t_i}}{5})^{2-2\gamma}$$
$$+5450(\frac{3X_{t_{i-1}}+3X_{t_i}}{5})^{2-2\gamma}+5675(\frac{4X_{t_{i-1}}+4X_{t_i}}{5})^{2-2\gamma}+1721(X_{t_i})^{2-2\gamma}\Big]\Big\}(t_i-t_{i-1})\Big]$$
$$-\Big[\Big\{\frac{1}{24192}\sum_{i=1}^{n}\Big[1471(X_{t_{i-1}})^{1-2\gamma}+6925(\frac{X_{t_{i-1}}+X_{t_i}}{5})^{1-2\gamma}+2950(\frac{2X_{t_{i-1}}+2X_{t_i}}{5})^{1-2\gamma}$$
$$+5450(\frac{3X_{t_{i-1}}+3X_{t_i}}{5})^{1-2\gamma}+5675(\frac{4X_{t_{i-1}}+4X_{t_i}}{5})^{1-2\gamma}+1721(X_{t_i})^{2}\Big]\Big\}(t_i-t_{i-1})\Big]$$
$$\Big[\Big\{\frac{1}{24192}\sum_{i=1}^{n}\Big[1471(X_{t_{i-1}})^{1-2\gamma}+6925(\frac{X_{t_{i-1}}+X_{t_i}}{5})^{1-2\gamma}+2950(\frac{2X_{t_{i-1}}+2X_{t_i}}{5})^{1-2\gamma}$$
$$+5450(\frac{3X_{t_{i-1}}+3X_{t_i}}{5})^{1-2\gamma}+5675(\frac{4X_{t_{i-1}}+4X_{t_i}}{5})^{1-2\gamma}+1721(X_{t_i})^{1-2\gamma}\Big]\Big\}(t_i-t_{i-1})\Big]\Big)^{-1},$$

$$\tilde{\beta}_{n,T,7}:=$$
$$\Big(\Big[\Big\{\frac{1}{24192}\sum_{i=1}^{n}\Big[1471(X_{t_{i-1}})^{-2\gamma}+6925(\frac{X_{t_{i-1}}+X_{t_i}}{5})^{-2\gamma}+2950(\frac{2X_{t_{i-1}}+2X_{t_i}}{5})^{-2\gamma}$$
$$+5450(\frac{3X_{t_{i-1}}+3X_{t_i}}{5})^{-2\gamma}+5675(\frac{4X_{t_{i-1}}+4X_{t_i}}{5})^{-2\gamma}+1721(X_{t_i})^{-2\gamma}\Big]\Big\}(X_{t_i}-X_{t_{i-1}})$$
$$+\gamma\Big\{\frac{1}{24192}\sum_{i=1}^{n}\Big[1471(X_{t_{i-1}})^{-1}+6925(\frac{X_{t_{i-1}}+X_{t_i}}{5})^{-1}+2950(\frac{2X_{t_{i-1}}+2X_{t_i}}{5})^{-1}$$
$$+5450(\frac{3X_{t_{i-1}}+3X_{t_i}}{5})^{-1}+5675(\frac{4X_{t_{i-1}}+4X_{t_i}}{5})^{-1}+1721(X_{t_i})^{-1}\Big]\Big\}(t_i-t_{i-1})\Big]$$
$$\Big[\Big\{\frac{1}{24192}\sum_{i=1}^{n}\Big[1471(X_{t_{i-1}})^{2-2\gamma}+6925(\frac{X_{t_{i-1}}+X_{t_i}}{5})^{2-2\gamma}+2950(\frac{2X_{t_{i-1}}+2X_{t_i}}{5})^{2-2\gamma}$$
$$+5450(\frac{3X_{t_{i-1}}+3X_{t_i}}{5})^{2-2\gamma}+5675(\frac{4X_{t_{i-1}}+4X_{t_i}}{5})^{2-2\gamma}+1721(X_{t_i})^{2-2\gamma}\Big]\Big\}(t_i-t_{i-1})\Big]$$

$$
\begin{aligned}
&- \left[\left\{\frac{1}{24192}\sum_{i=1}^{n}\left[1471(X_{t_{i-1}})^{1-2\gamma}+6925(\frac{X_{t_{i-1}}+X_{t_i}}{5})^{1-2\gamma}+2950(\frac{2X_{t_{i-1}}+2X_{t_i}}{5})^{1-2\gamma}\right.\right.\\
&\left.\left.+5450(\frac{3X_{t_{i-1}}+3X_{t_i}}{5})^{1-2\gamma}+5675(\frac{4X_{t_{i-1}}+4X_{t_i}}{5})^{1-2\gamma}+1721(X_{t_i})^{1-2\gamma}\right]\right\}(X_{t_i}-X_{t_{i-1}})-\frac{1-2\gamma}{2}T\Bigg]\\
&\left[\left\{\frac{1}{24192}\sum_{i=1}^{n}\left[1471(X_{t_{i-1}})^{1-2\gamma}+6925(\frac{X_{t_{i-1}}+X_{t_i}}{5})^{1-2\gamma}+2950(\frac{2X_{t_{i-1}}+2X_{t_i}}{5})^{1-2\gamma}\right.\right.\right.\\
&\left.\left.\left.+5450(\frac{3X_{t_{i-1}}+3X_{t_i}}{5})^{1-2\gamma}+5675(\frac{4X_{t_{i-1}}+4X_{t_i}}{5})^{1-2\gamma}+1721(X_{t_i})^{1-2\gamma}\right]\right\}(t_i-t_{i-1})\right]\Bigg)\\
&\Bigg(\Bigg[\left\{\frac{1}{24192}\sum_{i=1}^{n}\left[1471(X_{t_{i-1}})^{-2\gamma}+6925(\frac{X_{t_{i-1}}+X_{t_i}}{5})^{-2\gamma}+2950(\frac{2X_{t_{i-1}}+2X_{t_i}}{5})^{-2\gamma}\right.\right.\\
&\left.\left.+5450(\frac{3X_{t_{i-1}}+3X_{t_i}}{5})^{-2\gamma}+5675(\frac{4X_{t_{i-1}}+4X_{t_i}}{5})^{-2\gamma}+1721(X_{t_i})^{2}\right]\right\}(t_i-t_{i-1})\Bigg]\\
&+\gamma\left[\left\{\frac{1}{24192}\sum_{i=1}^{n}\left[1471(X_{t_{i-1}})^{-1}+6925(\frac{X_{t_{i-1}}+X_{t_i}}{5})^{-1}+2950(\frac{2X_{t_{i-1}}+2X_{t_i}}{5})^{-1}\right.\right.\right.\\
&\left.\left.\left.+5450(\frac{3X_{t_{i-1}}+3X_{t_i}}{5})^{2}+5675(\frac{4X_{t_{i-1}}+4X_{t_i}}{5})^{-1}+1721(X_{t_i})^{-1}\right]\right\}(t_i-t_{i-1})\right]\\
&- \left[\left\{\frac{1}{24192}\sum_{i=1}^{n}\left[1471(X_{t_{i-1}})^{1-2\gamma}+6925(\frac{X_{t_{i-1}}+X_{t_i}}{5})^{1-2\gamma}+2950(\frac{2X_{t_{i-1}}+2X_{t_i}}{5})^{1-2\gamma}\right.\right.\\
&\left.\left.+5450(\frac{3X_{t_{i-1}}+3X_{t_i}}{5})^{1-2\gamma}+5675(\frac{4X_{t_{i-1}}+4X_{t_i}}{5})^{1-2\gamma}+1721(X_{t_i})^{1-2\gamma}\right]\right\}(X_{t_i}-X_{t_{i-1}})-\frac{1-2\gamma}{2}T\Bigg]\\
&\left[\left\{\frac{1}{24192}\sum_{i=1}^{n}\left[1471(X_{t_{i-1}})^{-2\gamma}+6925(\frac{X_{t_{i-1}}+X_{t_i}}{5})^{-2\gamma}+2950(\frac{2X_{t_{i-1}}+2X_{t_i}}{5})^{-2\gamma}\right.\right.\right.\\
&\left.\left.\left.+5450(\frac{3X_{t_{i-1}}+3X_{t_i}}{5})^{-2\gamma}+5675(\frac{4X_{t_{i-1}}+4X_{t_i}}{5})^{-2\gamma}+1721(X_{t_i})^{-2\gamma}\right]\right\}(t_i-t_{i-1})\right]\Bigg)^{-1}
\end{aligned}
$$

$$
\begin{aligned}
&\tilde{\alpha}_{n,T,8} :=\\
&\Bigg(\Bigg[\left\{\frac{1}{90}\sum_{i=1}^{n}\left[7(X_{t_{i-1}})^{-2\gamma}+32(\frac{3X_{t_{i-1}}+X_{t_i}}{4})^{-2\gamma}+12(\frac{X_{t_{i-1}}+X_{t_i}}{2})^{-2\gamma}\right.\right.\\
&\left.\left.+32(\frac{X_{t_{i-1}}+3X_{t_i}}{4})^{-2\gamma}+7(X_{t_i})^{-2\gamma}\right]\right\}(X_{t_i}-X_{t_{i-1}})\\
&+\gamma\left\{\frac{1}{90}\sum_{i=1}^{n}\left[7(X_{t_{i-1}})^{-1}+32(\frac{3X_{t_{i-1}}+X_{t_i}}{4})^{-1}+12(\frac{X_{t_{i-1}}+X_{t_i}}{2})^{-1}\right.\right.\\
&\left.\left.+32(\frac{X_{t_{i-1}}+3X_{t_i}}{4})^{-1}+7(X_{t_i})^{\gamma}\right]\right\}(t_i-t_{i-1})\Bigg]\\
&\left[\left\{\frac{1}{90}\sum_{i=1}^{n}\left[7(X_{t_{i-1}})^{1-2\gamma}+32(\frac{3X_{t_{i-1}}+X_{t_i}}{4})^{1-2\gamma}+12(\frac{X_{t_{i-1}}+X_{t_i}}{2})^{1-2\gamma}\right.\right.\right.\\
&\left.\left.\left.+32(\frac{X_{t_{i-1}}+3X_{t_i}}{4})^{1-2\gamma}+7(X_{t_i})^{1-2\gamma}\right]\right\}(t_i-t_{i-1})\right]\\
&- \left[\left\{\frac{1}{90}\sum_{i=1}^{n}\left[7(X_{t_{i-1}})^{1-2\gamma}+32(\frac{3X_{t_{i-1}}+X_{t_i}}{4})^{1-2\gamma}+12(\frac{X_{t_{i-1}}+X_{t_i}}{2})^{1-2\gamma}\right.\right.\right.\\
&\left.\left.\left.+32(\frac{X_{t_{i-1}}+3X_{t_i}}{4})^{1-2\gamma}+7(t_i,X_{t_i})^{1-2\gamma}\right]\right\}(X_{t_i}-X_{t_{i-1}})-\frac{1-2\gamma}{2}T\Bigg]\\
&\left[\left\{\frac{1}{90}\sum_{i=1}^{n}\left[7(X_{t_{i-1}})^{-2\gamma}+32(\frac{3X_{t_{i-1}}+X_{t_i}}{4})^{-2\gamma}+12(\frac{X_{t_{i-1}}+X_{t_i}}{2})^{-2\gamma}\right.\right.\right.\\
&\left.\left.\left.+32(\frac{X_{t_{i-1}}+3X_{t_i}}{4})^{-2\gamma}+7(X_{t_i})^{-2\gamma}\right]\right\}(t_i-t_{i-1})\right]\Bigg)\\
&\Bigg(\Bigg[\left\{\frac{1}{90}\sum_{i=1}^{n}\left[7(X_{t_{i-1}})^{-2\gamma}+32(\frac{3X_{t_{i-1}}+X_{t_i}}{4})^{-2\gamma}+12(\frac{X_{t_{i-1}}+X_{t_i}}{2})^{-2\gamma}\right.\right.\\
&\left.\left.+32(\frac{X_{t_{i-1}}+3X_{t_i}}{4})^{-2\gamma}+7(X_{t_i})^{-2\gamma}\right]\right\}(t_i-t_{i-1})\Bigg]
\end{aligned}
$$

$$\left[\left\{\frac{1}{90}\sum_{i=1}^{n}\left[7(X_{t_{i-1}})^{2-2\gamma}+32(\frac{3X_{t_{i-1}}+X_{t_i}}{4})^{2-2\gamma}+12(\frac{X_{t_{i-1}}+X_{t_i}}{2})^{2-2\gamma}\right.\right.\right.$$
$$\left.\left.\left.+32(\frac{X_{t_{i-1}}+3X_{t_i}}{4})^{2-2\gamma}+7(X_{t_i})^{-1}\right]\right\}(t_i-t_{i-1})\right]$$
$$-\left[\left\{\frac{1}{90}\sum_{i=1}^{n}\left[7(X_{t_{i-1}})^{1-2\gamma}+32(\frac{3X_{t_{i-1}}+X_{t_i}}{4})^{1-2\gamma}+12(\frac{X_{t_{i-1}}+X_{t_i}}{2})^{1-2\gamma}\right.\right.\right.$$
$$\left.\left.\left.+32(\frac{X_{t_{i-1}}+3X_{t_i}}{4})^{1-2\gamma}+7(X_{t_i})^{1-2\gamma}\right]\right\}(t_i-t_{i-1})\right]$$
$$\left[\left\{\frac{1}{90}\sum_{i=1}^{n}\left[7(X_{t_{i-1}})^{1-2\gamma}+32(\frac{3X_{t_{i-1}}+X_{t_i}}{4})^{1-2\gamma}+12(\frac{X_{t_{i-1}}+X_{t_i}}{2})^{1-2\gamma}\right.\right.\right.$$
$$\left.\left.\left.+32(\frac{X_{t_{i-1}}+3X_{t_i}}{4})^{1-2\gamma}+7(X_{t_i})^{1-2\gamma}\right]\right\}(t_i-t_{i-1})\right]\Bigg)^{-1},$$

$$\tilde{\beta}_{n,T,8} := \Bigg(\Bigg[\left\{\frac{1}{90}\sum_{i=1}^{n}\left[7(X_{t_{i-1}})^{-2\gamma}+32(\frac{3X_{t_{i-1}}+X_{t_i}}{4})^{-2\gamma}+12(\frac{X_{t_{i-1}}+X_{t_i}}{2})^{-2\gamma}\right.\right.$$
$$\left.\left.+32(\frac{X_{t_{i-1}}+3X_{t_i}}{4})^{-2\gamma}+7(X_{t_i})^{-2\gamma}\right]\right\}(X_{t_i}-X_{t_{i-1}})$$
$$+\gamma\left\{\frac{1}{90}\sum_{i=1}^{n}\left[7(X_{t_{i-1}})^{-1}+32(\frac{3X_{t_{i-1}}+X_{t_i}}{4})^{-1}+12(\frac{X_{t_{i-1}}+X_{t_i}}{2})^{-1}\right.\right.$$
$$\left.\left.+32(\frac{X_{t_{i-1}}+3X_{t_i}}{4})^{-1}+7(X_{t_i})^{\gamma}\right]\right\}(t_i-t_{i-1})\Bigg]$$
$$\left[\left\{\frac{1}{90}\sum_{i=1}^{n}\left[7(X_{t_{i-1}})^{2-2\gamma}+32(\frac{3X_{t_{i-1}}+X_{t_i}}{4})^{2-2\gamma}+12(\frac{X_{t_{i-1}}+X_{t_i}}{2})^{2-2\gamma}\right.\right.\right.$$
$$\left.\left.\left.+32(\frac{X_{t_{i-1}}+3X_{t_i}}{4})^{2-2\gamma}+7(X_{t_i})^{2-2\gamma}\right]\right\}(t_i-t_{i-1})\right]$$
$$-\left[\left\{\frac{1}{90}\sum_{i=1}^{n}\left[7(X_{t_{i-1}})^{1-2\gamma}+32(\frac{3X_{t_{i-1}}+X_{t_i}}{4})^{1-2\gamma}+12(\frac{X_{t_{i-1}}+X_{t_i}}{2})^{1-2\gamma}\right.\right.\right.$$
$$\left.\left.\left.+32(\frac{X_{t_{i-1}}+3X_{t_i}}{4})^{1-2\gamma}+7(t_i,X_{t_i})^{1-2\gamma}\right]\right\}(X_{t_i}-X_{t_{i-1}})-\frac{1-2\gamma}{2}T\right]$$
$$\left[\left\{\frac{1}{90}\sum_{i=1}^{n}\left[7(X_{t_{i-1}})^{1-2\gamma}+32(\frac{3X_{t_{i-1}}+X_{t_i}}{4})^{1-2\gamma}+12(\frac{X_{t_{i-1}}+X_{t_i}}{2})^{1-2\gamma}\right.\right.\right.$$
$$\left.\left.\left.+32(\frac{X_{t_{i-1}}+3X_{t_i}}{4})^{1-2\gamma}+7(X_{t_i})^{1-2\gamma}\right]\right\}(t_i-t_{i-1})\right]\Bigg)$$
$$\Bigg(\Bigg[\left\{\frac{1}{90}\sum_{i=1}^{n}\left[7(X_{t_{i-1}})^{-2\gamma}+32(\frac{3X_{t_{i-1}}+X_{t_i}}{4})^{-2\gamma}+12(\frac{X_{t_{i-1}}+X_{t_i}}{2})^{-2\gamma}\right.\right.$$
$$\left.\left.+32(\frac{X_{t_{i-1}}+3X_{t_i}}{4})^{-2\gamma}+7(X_{t_i})^{-2\gamma}\right]\right\}(t_i-t_{i-1})$$
$$+\gamma\left\{\frac{1}{90}\sum_{i=1}^{n}\left[7(X_{t_{i-1}})^{-1}+32(\frac{3X_{t_{i-1}}+X_{t_i}}{4})^{-1}+12(\frac{X_{t_{i-1}}+X_{t_i}}{2})^{-1}\right.\right.$$
$$\left.\left.+32(\frac{X_{t_{i-1}}+3X_{t_i}}{4})^{-1}+7(X_{t_i})^{-1}\right]\right\}(t_i-t_{i-1})\Bigg]$$
$$\left[\left\{\frac{1}{90}\sum_{i=1}^{n}\left[7(X_{t_{i-1}})^{1-2\gamma}+32(\frac{3X_{t_{i-1}}+X_{t_i}}{4})^{1-2\gamma}+12(\frac{X_{t_{i-1}}+X_{t_i}}{2})^{1-2\gamma}\right.\right.\right.$$

$$+32(\frac{X_{t_{i-1}}+3X_{t_i}}{4})^{1-2\gamma}+7(X_{t_i})^{1-2\gamma}\Big]\Big\}(t_i-t_{i-1})\Big]$$
$$-\Big[\Big\{\frac{1}{90}\sum_{i=1}^{n}\Big[7(X_{t_{i-1}})^{1-2\gamma}+32(\frac{3X_{t_{i-1}}+X_{t_i}}{4})^{1-2\gamma}+12(\frac{X_{t_{i-1}}+X_{t_i}}{2})^{1-2\gamma}$$
$$+32(\frac{X_{t_{i-1}}+3X_{t_i}}{4})^{1-2\gamma}+7(X_{t_i})^{1-2\gamma}\Big]\Big\}(t_i-t_{i-1})-\frac{1-2\gamma}{2}T\Big]$$
$$\Big[\Big\{\frac{1}{90}\sum_{i=1}^{n}\Big[7(X_{t_{i-1}})^{-2\gamma}+32(\frac{3X_{t_{i-1}}+X_{t_i}}{4})^{-2\gamma}+12(\frac{X_{t_{i-1}}+X_{t_i}}{2})^{-2\gamma}$$
$$+32(\frac{X_{t_{i-1}}+3X_{t_i}}{4})^{-2\gamma}+7(X_{t_i})^{-2\gamma}\Big]\Big\}(t_i-t_{i-1})\Big]\Big)^{-1},$$

$\tilde{\alpha}_{n,T,9} :=$
$$\Big(\Big[\Big\{\frac{1}{288}\sum_{i=1}^{n}\Big[19(X_{t_{i-1}})^{-2\gamma}+75(\frac{4X_{t_{i-1}}+X_{t_i}}{5})^{-2\gamma}+50(\frac{3X_{t_{i-1}}+2X_{t_i}}{5})^{-2\gamma}$$
$$+50(\frac{2X_{t_{i-1}}+3X_{t_i}}{5})^{-2\gamma}+75(\frac{X_{t_{i-1}}+4X_{t_i}}{5})^{1-2\gamma}+19(X_{t_i})^{-2\gamma}\Big]\Big\}(X_{t_i}-X_{t_{i-1}})$$
$$+\gamma\Big[\Big\{\frac{1}{288}\sum_{i=1}^{n}\Big[19(X_{t_{i-1}})^{-1}+75(\frac{4X_{t_{i-1}}+X_{t_i}}{5})^{-1}+50(\frac{3X_{t_{i-1}}+2X_{t_i}}{5})^{-1}$$
$$+50(\frac{2X_{t_{i-1}}+3X_{t_i}}{5})^{-1}+75(\frac{X_{t_{i-1}}+4X_{t_i}}{5})^{-1}+19(X_{t_i})^{-1}\Big]\Big\}(t_i-t_{i-1})\Big]$$
$$\Big[\Big[\Big\{\frac{1}{288}\sum_{i=1}^{n}\Big[19(X_{t_{i-1}})^{1-2\gamma}+75(\frac{4X_{t_{i-1}}+X_{t_i}}{5})^{1-2\gamma}+50(\frac{3X_{t_{i-1}}+2X_{t_i}}{5})^{1-2\gamma}$$
$$+50(\frac{2X_{t_{i-1}}+3X_{t_i}}{5})^{1-2\gamma}+75(\frac{X_{t_{i-1}}+4X_{t_i}}{5})^{2-2\gamma}+19(X_{t_i})^{1-2\gamma}\Big]\Big\}(t_i-t_{i-1})\Big]$$
$$-\Big[\Big[\Big\{\frac{1}{288}\sum_{i=1}^{n}\Big[19(X_{t_{i-1}})^{1-2\gamma}+75(\frac{4X_{t_{i-1}}+X_{t_i}}{5})^{1-2\gamma}+50(\frac{3X_{t_{i-1}}+2X_{t_i}}{5})^{1-2\gamma}$$
$$+50(\frac{2X_{t_{i-1}}+3X_{t_i}}{5})^{1-2\gamma}+75(\frac{X_{t_{i-1}}+4X_{t_i}}{5})^{2-2\gamma}+19(X_{t_i})^{1-2\gamma}\Big]\Big\}(X_{t_i}-X_{t_{i-1}})-\frac{1-2\gamma}{2}T\Big]$$
$$\Big[\Big\{\frac{1}{288}\sum_{i=1}^{n}\Big[19(X_{t_{i-1}})^{-2\gamma}+75(\frac{4X_{t_{i-1}}+X_{t_i}}{5})^{-2\gamma}+50(\frac{3X_{t_{i-1}}+2X_{t_i}}{5})^{-2\gamma}$$
$$+50(\frac{2X_{t_{i-1}}+3X_{t_i}}{5})^{-2\gamma}+75(\frac{X_{t_{i-1}}+4X_{t_i}}{5})^{-2\gamma}+19(X_{t_i})^{-2\gamma}\Big]\Big\}(X_{t_i}-X_{t_{i-1}})\Big)$$
$$\Big(\Big[\Big\{\frac{1}{288}\sum_{i=1}^{n}\Big[19(X_{t_{i-1}})^{-2\gamma}+75(\frac{4X_{t_{i-1}}+X_{t_i}}{5})^{-2\gamma}+50(\frac{3X_{t_{i-1}}+2X_{t_i}}{5})^{-2\gamma}$$
$$+50(\frac{2X_{t_{i-1}}+3X_{t_i}}{5})^{-2\gamma}+75(\frac{X_{t_{i-1}}+4X_{t_i}}{5})^{\gamma}+19(X_{t_i})^{-2\gamma}\Big]\Big\}(t_i-t_{i-1})\Big]$$
$$\Big[\Big[\Big\{\frac{1}{288}\sum_{i=1}^{n}\Big[19(X_{t_{i-1}})^{2-2\gamma}+75(\frac{4X_{t_{i-1}}+X_{t_i}}{5})^{2-2\gamma}+50(\frac{3X_{t_{i-1}}+2X_{t_i}}{5})^{2-2\gamma}$$
$$+50(\frac{2X_{t_{i-1}}+3X_{t_i}}{5})^{2-2\gamma}+75(\frac{X_{t_{i-1}}+4X_{t_i}}{5})^{2-2\gamma}+19(X_{t_i})^{2-2\gamma}\Big]\Big\}(t_i-t_{i-1})\Big]$$

$$- \left[\left[\left\{\frac{1}{288}\sum_{i=1}^{n}\left[19(X_{t_{i-1}})^{1-2\gamma}+75(\frac{4X_{t_{i-1}}+X_{t_i}}{5})^{1-2\gamma}+50(\frac{3X_{t_{i-1}}+2X_{t_i}}{5})^{1-2\gamma}\right.\right.\right.$$
$$\left.\left.\left.+50(\frac{2X_{t_{i-1}}+3X_{t_i}}{5})^{1-2\gamma}+75(\frac{X_{t_{i-1}}+4X_{t_i}}{5})^{1-2\gamma}+19(X_{t_i})^{1-2\gamma}\right]\right\}(t_i-t_{i-1})\right]^2\Bigg)^{-1},$$

$$\tilde{\beta}_{n,T,9} := \Bigg(\left[\left\{\frac{1}{288}\sum_{i=1}^{n}\left[19(X_{t_{i-1}})^{-2\gamma}+75(\frac{4X_{t_{i-1}}+X_{t_i}}{5})^{-2\gamma}+50(\frac{3X_{t_{i-1}}+2X_{t_i}}{5})^{-2\gamma}\right.\right.\right.$$
$$\left.\left.\left.+50(\frac{2X_{t_{i-1}}+3X_{t_i}}{5})^{-2\gamma}+75(\frac{X_{t_{i-1}}+4X_{t_i}}{5})^{1-2\gamma}+19(X_{t_i})^{-2\gamma}\right]\right\}(X_{t_i}-X_{t_{i-1}})$$
$$+\gamma\left[\left\{\frac{1}{288}\sum_{i=1}^{n}\left[19(X_{t_{i-1}})^{-1}+75(\frac{4X_{t_{i-1}}+X_{t_i}}{5})^{-1}+50(\frac{3X_{t_{i-1}}+2X_{t_i}}{5})^{-1}\right.\right.\right.$$
$$\left.\left.\left.+50(\frac{2X_{t_{i-1}}+3X_{t_i}}{5})^{-1}+75(\frac{X_{t_{i-1}}+4X_{t_i}}{5})^{-1}+19(X_{t_i})^{-1}\right]\right\}(t_i-t_{i-1})\right]$$
$$\left[\left[\left\{\frac{1}{288}\sum_{i=1}^{n}\left[19(X_{t_{i-1}})^{2-2\gamma}+75(\frac{4X_{t_{i-1}}+X_{t_i}}{5})^{2-2\gamma}+50(\frac{3X_{t_{i-1}}+2X_{t_i}}{5})^{2-2\gamma}\right.\right.\right.\right.$$
$$\left.\left.\left.\left.+50(\frac{2X_{t_{i-1}}+3X_{t_i}}{5})^{2-2\gamma}+75(\frac{X_{t_{i-1}}+4X_{t_i}}{5})^{2-2\gamma}+19(X_{t_i})^{2-2\gamma}\right]\right\}(t_i-t_{i-1})\right]$$
$$-\left[\left[\left\{\frac{1}{288}\sum_{i=1}^{n}\left[19(X_{t_{i-1}})^{1-2\gamma}+75(\frac{4X_{t_{i-1}}+X_{t_i}}{5})^{1-2\gamma}+50(\frac{3X_{t_{i-1}}+2X_{t_i}}{5})^{1-2\gamma}\right.\right.\right.\right.$$
$$\left.\left.\left.\left.+50(\frac{2X_{t_{i-1}}+3X_{t_i}}{5})^{1-2\gamma}+75(\frac{X_{t_{i-1}}+4X_{t_i}}{5})^{2-2\gamma}+19(X_{t_i})^{1-2\gamma}\right]\right\}(X_{t_i}-X_{t_{i-1}})-\frac{1-2\gamma}{2}T\right]$$
$$\left[\left\{\frac{1}{288}\sum_{i=1}^{n}\left[19(X_{t_{i-1}})^{-2\gamma}+75(\frac{4X_{t_{i-1}}+X_{t_i}}{5})^{1-2\gamma}+50(\frac{3X_{t_{i-1}}+2X_{t_i}}{5})^{1-2\gamma}\right.\right.\right.$$
$$\left.\left.\left.+50(\frac{2X_{t_{i-1}}+3X_{t_i}}{5})^{1-2\gamma}+75(\frac{X_{t_{i-1}}+4X_{t_i}}{5})^{1-2\gamma}+19(X_{t_i})^{1-2\gamma}\right]\right\}(X_{t_i}-X_{t_{i-1}})\right)$$
$$\Bigg(\left[\left\{\frac{1}{288}\sum_{i=1}^{n}\left[19(X_{t_{i-1}})^{-2\gamma}+75(\frac{4X_{t_{i-1}}+X_{t_i}}{5})^{-2\gamma}+50(\frac{3X_{t_{i-1}}+2X_{t_i}}{5})^{-2\gamma}\right.\right.\right.$$
$$\left.\left.\left.+50(\frac{2X_{t_{i-1}}+3X_{t_i}}{5})^{-2\gamma}+75(\frac{X_{t_{i-1}}+4X_{t_i}}{5})^{\gamma}+19(X_{t_i})^{-2\gamma}\right]\right\}(X_{t_i}-X_{t_{i-1}})\right]$$
$$+\gamma\left[\left\{\frac{1}{288}\sum_{i=1}^{n}\left[19(X_{t_{i-1}})^{-1}+75(\frac{4X_{t_{i-1}}+X_{t_i}}{5})^{-1}+50(\frac{3X_{t_{i-1}}+2X_{t_i}}{5})^{-1}\right.\right.\right.$$
$$\left.\left.\left.+50(\frac{2X_{t_{i-1}}+3X_{t_i}}{5})^{-1}+75(\frac{X_{t_{i-1}}+4X_{t_i}}{5})^{-1}+19(X_{t_i})^{-1}\right]\right\}(t_i-t_{i-1})\right]$$
$$\left[\left[\left\{\frac{1}{288}\sum_{i=1}^{n}\left[19(X_{t_{i-1}})^{2-2\gamma}+75(\frac{4X_{t_{i-1}}+X_{t_i}}{5})^{1-2\gamma}+50(\frac{3X_{t_{i-1}}+2X_{t_i}}{5})^{1-2\gamma}\right.\right.\right.\right.$$
$$\left.\left.\left.\left.+50(\frac{2X_{t_{i-1}}+3X_{t_i}}{5})^{1-2\gamma}+75(\frac{X_{t_{i-1}}+4X_{t_i}}{5})^{2-2\gamma}+19(X_{t_i})^{1-2\gamma}\right]\right\}(t_i-t_{i-1})\right]$$
$$-\left[\left[\left\{\frac{1}{288}\sum_{i=1}^{n}\left[19(X_{t_{i-1}})^{1-2\gamma}+75(\frac{4X_{t_{i-1}}+X_{t_i}}{5})^{1-2\gamma}+50(\frac{3X_{t_{i-1}}+2X_{t_i}}{5})^{1-2\gamma}\right.\right.\right.\right.$$

$$+50(\frac{2X_{t_{i-1}}+3X_{t_i}}{5})^{1-2\gamma}+75(\frac{X_{t_{i-1}}+4X_{t_i}}{5})^{1-2\gamma}+19(X_{t_i})^{1-2\gamma}\Big]\Big\}(X_{t_i}-X_{t_{i-1}})-\frac{1-2\gamma}{2}T\Big]$$

$$\Big[\Big[\Big\{\frac{1}{288}\sum_{i=1}^{n}\Big[19(X_{t_{i-1}})^{1-2\gamma}+75(\frac{4X_{t_{i-1}}+X_{t_i}}{5})^{1-2\gamma}+50(\frac{3X_{t_{i-1}}+2X_{t_i}}{5})^{1-2\gamma}$$

$$+50(\frac{2X_{t_{i-1}}+3X_{t_i}}{5})^{1-2\gamma}+75(\frac{X_{t_{i-1}}+4X_{t_i}}{5})^{1-2\gamma}+19(X_{t_i})^{1-2\gamma}\Big]\Big\}(t_i-t_{i-1})\Big]\Big)^{-1}.$$

4 Test Function Estimator of Elasticity of Volatility and Stochastic Elasticity Model

We first discuss estimation of elasticity when it is a constant parameter. Generalized method of moments (GMM), which is a generalization of weighted least squares method with the random weight being the inverse of the covariance matrix, proposed by Hansen [26] is a popular estimation method in financial econometrics where likelihood may not be available, that is maximum likelihood estimation is not feasible. Also one may not need the distribution of the error term in the model. GMM estimators are in general consistent, asymptotic normal and asymptotically efficient. Conley et al. [27] proposed to estimate the elasticity parameter γ by minimizing a generalized method of moments (GMM) criterion function. The criterion function is based on a combined set of moment conditions constructed from the level and difference test functions, whereas the elasticity is treated as an unknown parameter to be estimated along with the drift parameters. To facilitate the interpretation of the GMM test statistics, we can estimate the elasticity parameter γ by the two-step GMM estimation procedure proposed in Conley et al. [27]. In the first step, we use our estimators of the drift parameters of the previous section as a function of the variance elasticity γ and plug them into the moment conditions formed from test functions of the first differences to estimate γ. Then we estimate γ by GMM method. Based on empirical data fitting to the model, the value of the elasticity parameter is known to be near 0.75.

Consider a test function φ in the domain of the generator $\mathcal{G}$ of the sub-fractional diffusion process X satisfying the sub-fractional SDE

$$dX_t=\mu(X_t)dt+\sigma(X_t)dW_t^H.$$

Since $E[\varphi(X_t)]$ is constant over time, it has zero derivative. We have

$$E[\mathcal{G}\varphi(X_t)]=E[\mu(X_t)\varphi'(X_t)+H(2H-1)\sigma^2(X_t)\varphi''(X_t)]=0.$$

An efficient test function would be l_T, the derivative of the log-likelihood. The resulting test function estimator using $E[\mathcal{G}l_T]=0$ will be efficient. It will be more efficient than the quasi-maximum likelihood estimator (QMLE) that uses the moment condition $E[l_T]=0$. In effect, the application of the generator to the score function adjusts the moment conditions optimally for the presence of temporal dependence.

One can use localized test functions by multiplying the first derivative by a smooth kernel K.

Non-homogeneous time-dependent elasticity of volatility has been proposed in Fan et al. [15]. We propose a sub-fractional stochastic elasticity of volatility model

$$dX_t=\alpha(\beta-X_t)dt+\sigma X_t^{\gamma_t}dW_t^H,$$

$$d\gamma_t=(\alpha_1+\beta_1\gamma_t)dt+\sigma_1\sqrt{\gamma_t}(\rho W_t^H+\sqrt{1-\rho^2}B_t^H)$$

where W_t^H and B_t^H are two independent sub-fractional Brownian motions and ρ is the correlation between the interest rate and the elasticity. In order to estimate γ_t and its parameters based on the interest rate data, one can use the stochastic filtering method as in Bishwal [28].

5 Conclusion

We proposed a stochastic hybrid asset price model where the volatility, the interest rate, the leverage and the elasticity are all stochastic. We proposed a new term structure model called the sub-fractional CKLS model by generalising the CKLS model which preserves medium range memory and whose increments are nonstationary. We proposed a new interest rate parsimonious model called the Generalized Ait-Sahalia (GAS) model with 19 parameters being driven by sub-fractional Brownan motion which preserves medium range memory and whose increments are nonstationary. We proposed a sub-fractional stochastic elasticity of volatility model. We obtained several new estimators of the drift parameters of the sub-fractional CKLS interest rate model using a new algorithm based on moment problem which would be useful for simulation, estimation, bond pricing and pricing interest rate derivatives. The new algorithm can be used for useful for estimation in other continuous time models where higher order approximations are necessary.

Competing Interests

Author has declared that no competing interests exist.

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Appendix: Newton-Cotes Distributions

Here we summarise the different distributions used to construct the estimators.

Itô Distribution
Support: $s_1 = 0$
Probability: $p_1 = 1$
Moment: $\mu_1 = 0$
Rate: $\nu = 1$

McKean Distribution
Support: $s_1 = 1$
Probability: $p_1 = 1$
Moment: $\mu_1 = 1$
Rate: $\nu = 1$

Distribution 1 (Fisk Distribution)
Support: $(s_1, s_2) = (0, 1)$
Probability: $(p_1, p_2) = (\frac{1}{2}, \frac{1}{2})$
Moment: $(\mu_1, \mu_2) = (\frac{1}{2}, \frac{1}{4})$
Rate: $\nu = 2$

Distribution 2 (Stratonovich Distribution)
Support: $s_1 = \frac{1}{2}$
Probability: $p_1 = 1$
Moment: $(\mu_1, \mu_2) = (\frac{1}{2}, \frac{1}{2})$
Rate: $\nu = 2$

Distribution 3
Support: $(s_1, s_2) = (0, \frac{2}{3})$
Probability: $(p_1, p_2) = (\frac{1}{4}, \frac{3}{4})$
Moment: $(\mu_1, \mu_2, \mu_3) = (\frac{1}{2}, \frac{1}{3}, \frac{2}{9})$
Rate: $\nu = 3$

Distribution 4
Support: $(s_1, s_2) = (\frac{1}{3}, 1)$
Probability: $(p_1, p_2) = \frac{3}{4}, \frac{1}{4}$
Moment: $(\mu_1, \mu_2, \mu_3) = (\frac{1}{2}, \frac{1}{3}, \frac{10}{36})$
Rate: $\nu = 3$

Distribution 5 (Simpson's 2/3 Distribution)
Support: $(s_1, s_2, s_3) = (0, \frac{1}{2}, 1)$
Probability: $(p_1, p_2, p_3) = (\frac{1}{6}, \frac{2}{3}, \frac{1}{6})$
Moment: $(\mu_1, \mu_2, \mu_3, \mu_4) = (\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{5}{24})$
Rate: $\nu = 4$

Distribution 6 (Simpson's 3/8 Distribution)
Support: $(s_1, s_2, s_3, s_4) = (0, \frac{1}{3}, \frac{2}{3}, 1)$
Probability: $(p_1, p_2, p_3, p_4) = (\frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8})$
Moment: $(\mu_1, \mu_2, \mu_3, \mu_4) = (\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{11}{54})$
Rate: $\nu = 4$

Distribution 7
Support: $(s_1, s_2, s_3, s_4, s_5) = (0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1)$
Probability: $(p_1, p_2, p_3, p_4, p_5) = (\frac{1471}{24192}, \frac{6925}{24192}, \frac{1475}{12096}, \frac{2725}{12096}, \frac{5675}{24192}, \frac{1721}{24192})$
Moment: $(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6) = (\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{841}{5040})$.
Rate: $\nu = 5$

Distribution 8 (Boole's Distribution)
Support: $(s_1, s_2, s_3, s_4, s_5) = (0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1)$
Probability: $(p_1, p_2, p_3, p_4, p_5) = (\frac{7}{90}, \frac{32}{90}, \frac{2}{15}, \frac{32}{90}, \frac{7}{90})$
Moment: $(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6) = (\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{110}{768})$
Rate: $\nu = 6$

Distribution 9
Support: $(s_1, s_2, s_3, s_4, s_5) = (0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1)$
Probability: $(p_1, p_2, p_3, p_4, p_5) = (\frac{19}{288}, \frac{75}{288}, \frac{50}{288}, \frac{50}{288}, \frac{75}{288}, \frac{19}{288})$
Moment: $(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6) = (\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{3219}{22500})$
Rate: $\nu = 6$

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