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# On Direct Fuzzy Stability of General Quartic Functional Equation

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***Author's contribution***

*The sole author designed, analyzed, interpreted and prepared the manuscript.*

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## Abstract

The goal of this paper is to investigate the Generalized Hyers - Ulam - Rassias (HUR) stability of general quartic functional equation (GQFE)

$$f_q(kr_1 + (k-1)r_2) + f_q(kr_1 - (k-1)r_2) = 2k^4 f_q(r_1) + 2(k-1)^4 f_q(r_2)$$
$$+6k^2(k-1)^2[f_q(r_1 + r_2) + f_q(r_1 - r_2)] - 12k^2(k-1)^2[f_q(r_1) + f_q(r_2)]$$

in fuzzy normed spaces (F.N. spaces). The stability of the equation is proved by using direct method. The stability in sense of Hyers - Ulam and Ulam- Gavruta - Rassias is also studied.

*Keywords: Generalised Hyers - Ulam - Rassias (HUR) stability; fuzzy normed space; general quartic functional equation.*

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# 1 Introduction

The theory of functional equations (FEs) is a vast area of nonlinear analysis, which is rather hard to explore. Geometry, economics, game theory, measure theory, dynamics, and a variety of other subjects all use FEs. In the subject of analysis, the study of solutions and stability results of FEs is a popular topic. The stability results are utilised to investigate additive mappings' asymptotic properties.

Next, let us recollect the chronicle in the stability theory for FEs. The concept of stability for various FEs arises when one replaces a FE by an inequality, which acts as a perturbation of the equation. The stability problem for the FEs about the stability of group homomorphisms was started by Ulam [1]. The Ulam′s question was to an extent solved by Hyers [2] in the case of approximately additive mappings. Thereafter, Hyers' result was generalized by Aoki [3] and improved for additive mappings, and subsequently improved by Rassias [4] for linear mappings by allowing the Cauchy difference to be unbounded. Subsequently, G ǎ vruta [5] generalized Rassias theorem and discussed the stability of linear FEs.

The QFE was first introduced by Rassias [6], who solved its Ulam stability problem. Later, Lee et al. [7] remodified Rassias' QFE and obtained its general solution. Numerous mathematicians have extensively studied the stability problems of various QFE in a variety of spaces, including intuitionistic fuzzy normed spaces, random normed spaces, non-Archimedean fuzzy normed spaces, Banach spaces and many other (see [8, 9, 10, 11]). Most of the proofs of stability problems in the sense of Hyers–Ulam have used Hyers' direct method. The exact solution of the FE is explicitly obtained as the limit of a sequence, which starts from the given approximate solution.

# 2 Preliminaries

Katsaras [12] explicated the postulation of the fuzzy norm over linear space. Since then, various mathematicians [13, 14, 15] gave the meaning of fuzzy norm over vector space from different perspectives.

**Definition 2.1.** [13] A function $\mathcal{M} : \mathcal{F}_b \times \mathbb{R} \to [0,1]$ ($\mathcal{F}_b$ being a real vector space over field F) is labeled as a *fuzzy norm* over $\mathcal{F}_b$ if, $\forall\, x, y \in \mathcal{F}_b$ and all $c, s \in \mathbb{R}, k \in F$:

1. $\mathcal{M}(x,c) = 0$ for $c \leq 0$,
2. x=0 iff $\mathcal{M}(x,c) = 1$ for all $c > 0$,
3. $\mathcal{M}(kx,c) = \mathcal{M}(x, \frac{c}{|k|})$ if k $\neq 0$,
4. $\mathcal{M}(x+y, s+c) \geq min\{\mathcal{M}(x,s), \mathcal{M}(y,c)\}$,
5. $\mathcal{M}(x,.)$ is non-decreasing function in $\mathbb{R}$ and $\lim_{c\to\infty} \mathcal{M}(x,c) = 1$,
6. $\mathcal{M}(x,.)$ is continuous on $\mathbb{R}$, $x \neq 0$.

The pair $(\mathcal{F}_b, \mathcal{M})$ is called as a *fuzzy normed vector space* (F.N. space)[13].

**Definition 2.2.** [13]

1. Let $(\mathcal{F}_b, \mathcal{M})$ be a F.N. space. A sequence $\{a_n\}$ in $\mathcal{F}_b$ is said to be *convergent* if $\exists$ an $a \in \mathcal{F}_b$ such that $\lim_{n\to\infty} \mathcal{M}(a_n - a, r) = 1$ for all $r > 0$, where $a$ is the limit of the sequence $\{a_n\}$, denoted by $\mathcal{M} - \lim_{n\to\infty} a_n = a$.
2. Let $(\mathcal{F}_b, \mathcal{M})$ be a F.N. space. A sequence $\{a_n\}$ in $\mathcal{F}_b$ is labeled as *Cauchy* if for each $\epsilon > 0$ and each $r > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $m > 0$, we have $\mathcal{M}(a_{n+m} - a_n, r) > 1 - \epsilon$.
3. The fuzzy norm is said to be *complete* if every Cauchy sequence is convergent and then the fuzzy normed vector space is called a *fuzzy Banach space.*

F. Skof [16] has studied HU stability of F.E.

$$f_q(x_1+x_2)+f_q(x_1-x_2)=2f_q(x_1)+2f_q(x_2)$$

for mapping $f_q$ from real normed space to Banach space. Under the same set-up, K. W. Jun et. al. [17] analyzed the stability problem of following cubic FE

$$f_c(2x_1+x_2)+f_c(2x_1-x_2)=2[f_c(x_1+x_2)+f_c(x_1-x_2)+6f_c(x_1)].$$

In this paper, we prove the Generalised HUR stability of the GQFE

$$\begin{aligned} f_q(kr_1+(k-1)r_2)+f_q(kr_1-(k-1)r_2) &= 2k^4 f_q(r_1)+2(k-1)^4 f_q(r_2) \\ &+6k^2(k-1)^2[f_q(r_1+r_2)+f_q(r_1-r_2)]-12k^2(k-1)^2[f_q(r_1)+f_q(r_2)] \end{aligned} \tag{2.1}$$

in F.N. spaces using direct method.

# 3 Generalized HUR Stability of the GQFE in Fuzzy Normed Spaces

Throughout this section, let $\mathcal{V}_s$ be a real linear space, and $(\mathcal{F}_n, \mathcal{M}')$ and $(\mathcal{F}_b, \mathcal{M})$ be a F.N. space and fuzzy Banach space respectively. Also, define

$$\begin{aligned} \hat{D}f_q(r_1,r_2) &= f_q(kr_1+(k-1)r_2)+f_q(kr_1-(k-1)r_2)-2k^4 f_q(r_1)-2(k-1)^4 f_q(r_2) \\ &-6k^2(k-1)^2[f_q(r_1+r_2)+f_q(r_1-r_2)]+12k^2(k-1)^2[f_q(r_1)+f_q(r_2)] \end{aligned}$$

for all $r_1, r_2 \in \mathcal{V}_s$ .

## 3.1 Theorem

Let $\in \{1,-1\}$ be fixed and $\eta : \mathcal{V}_s^2 \to \mathcal{F}_n$ be a function such that

$$\mathcal{M}'(\eta(kr_1,0),c) \geq \mathcal{M}'(p\eta(r_1,0),c) \tag{3.1}$$

for all $r_1 \in \mathcal{V}_s$ and all $c>0$, where $p>0$ with $\left(\frac{p}{k^4}\right)<1$, and

$$\lim_{b\to\infty} \mathcal{M}'(\eta(k^b r_1, k^b r_2), k^{4b}c)=1$$

for all $r_1, r_2 \in \mathcal{V}_s$ and all $c>0$. Let $f_q : \mathcal{V}_s \to \mathcal{F}_b$ be a mapping which maps zero to zero and satisfies

$$\mathcal{M}(\hat{D}f_q(r_1,r_2),c) \geq \mathcal{M}'(\eta(r_1,r_2),c) \tag{3.2}$$

for all $r_1, r_2 \in \mathcal{V}_s$ and all $c>0$. Then the limit

$$Q_n(r_1) = \mathcal{M} - \lim_{b\to\infty} \frac{1}{k^{4b}} f_q(k^{4b} r_1)$$

exist for all $r_1 \in \mathcal{V}_s$ with a unique quartic mapping $Q_n : \mathcal{V}_s \to \mathcal{F}_b$ such that

$$\mathcal{M}(f_q(r_1)-Q_n(r_1),c) \geq \mathcal{M}'(\eta(r_1,0), \frac{c|k^4-p|}{2}) \tag{3.3}$$

for all $r_1 \in \mathcal{V}_s$ and all $c>0$.

*Proof.* Let j= 1. Replace $(r_1, r_2)$ by $(r_1, 0)$ in (3.2) to get

$$\mathcal{M}(2f_q(kr_1) - 2k^4 f_q(r_1), c) \geq \mathcal{M}'(\eta(r_1, 0), c),$$

for all $r_1 \in \mathcal{V}_s$ and all $c > 0$. Or we can say,

$$\mathcal{M}\Big(f_q(kr_1) - k^4 f_q(r_1), \frac{c}{2}\Big) \geq \mathcal{M}'(\eta(r_1, 0), c). \tag{3.4}$$

After replacing $r_1$ by $k^b r_1$ in (3.4) we get,

$$\mathcal{M}\Big(\frac{f_q(k^{b+1} r_1)}{k^4} - f_q(k^b r_1), \frac{c}{2k^4}\Big) \geq \mathcal{M}'(\eta(k^b r_1, 0), c), \tag{3.5}$$

for all $r_1 \in \mathcal{V}_s$ and all $c > 0$. Using (3.1), we obtain

$$\mathcal{M}\Big(\frac{f_q(k^{b+1} r_1)}{k^4} - f_q(k^b r_1), \frac{c}{2k^4}\Big) \geq \mathcal{M}'\Big(\eta(r_1, 0), \frac{c}{p^b}\Big), \tag{3.6}$$

for all $r_1 \in \mathcal{V}_s$ and all $c > 0$. Replacing $c$ by $p^b c$ in (3.6), we get

$$\mathcal{M}\Big(\frac{f_q(k^{b+1} r_1)}{k^{4(b+1)}} - \frac{f_q(k^b r_1)}{k^{4b}}, \frac{p^b c}{2k^{4(b+1)}}\Big) \geq \mathcal{M}'(\eta(r_1, 0), c), \tag{3.7}$$

for all $r_1 \in \mathcal{V}_s$ and all $c > 0$. Next, we have

$$\frac{f_q(k^b r_1)}{k^{4b}} - f_q(r_1) = \sum_{j=0}^{b-1}\Big(\frac{f_q(k^{j+1} r_1)}{k^{4(j+1)}} - \frac{f_q(k^j r_1)}{k^{4j}}\Big).$$

Hence, from above equation and (3.7) we get,

$$\begin{aligned}
\mathcal{M}\Big(\frac{f_q(k^b r_1)}{k^{4b}} &- f_q(r_1), \sum_{j=0}^{b-1} \frac{p^j c}{2k^{4(j+1)}}\Big) \\
&\geq min\Big\{\mathcal{M}\Big(\frac{f_q(k^{j+1} r_1)}{k^{4(j+1)}} - \frac{f_q(k^j r_1)}{k^{4j}}, \frac{p^j c}{2k^{4(j+1)}}\Big) : j = 0, 1, ..., b-1\Big\} \\
&\geq \mathcal{M}'(\eta(r_1, 0), c),
\end{aligned} \tag{3.8}$$

for all $r_1 \in \mathcal{V}_s$ and all $c > 0$. Replacing $r_1$ by $k^a r_1$ in (3.8), we get

$$\begin{aligned}
\mathcal{M}\Big(\frac{f_q(k^{b+a} r_1)}{k^{4(b+a)}} - \frac{f_q(k^a r_1)}{k^{4a}}, \sum_{j=0}^{b-1} \frac{p^j c}{2k^{4(j+a+1)}}\Big) &\geq \mathcal{M}'(\eta(k^a r_1, 0), c) \\
&\geq \mathcal{M}'\Big(\eta(r_1, 0), \frac{c}{p^a}\Big).
\end{aligned} \tag{3.9}$$

Or simply,

$$\mathcal{M}\Big(\frac{f_q(k^{b+a} r_1)}{k^{4(b+a)}} - \frac{f_q(k^a r_1)}{k^{4a}}, \sum_{j=a}^{b+a-1} \frac{p^j c}{2k^{4(j+1)}}\Big) \geq \mathcal{M}'(\eta(r_1, 0), c), \tag{3.10}$$

for all $r_1 \in \mathcal{V}_s$ , $c > 0$ and all $a, b \geq 0$. Again replacing $c$ by $\frac{c}{\sum_{j=a}^{b+a-1} \frac{p^j}{2k^{4(j+1)}}}$ in (3.10), we get,

$$\mathcal{M}\Big(\frac{f_q(k^{b+a} r_1)}{k^{4(b+a)}} - \frac{f_q(k^a r_1)}{k^{4a}}, c\Big) \geq \mathcal{M}'\Big(\eta(r_1, 0), \frac{c}{\sum_{j=a}^{b+a-1} \frac{p^j}{2k^{4(j+1)}}}\Big), \tag{3.11}$$

for all $r_1 \in \mathcal{V}_s$, $c > 0$ and all $a, b \geq 0$. Since $0 < p < k^4$ and $\sum_{j=0}^{\infty} \left(\frac{p}{k^4}\right)^j < \infty$, the cauchy criterion for convergence and definition of *fuzzy normed space* implies that $\left\{\frac{f_q(k^b r_1)}{k^{4b}}\right\}$ is a Cauchy sequence in $(\mathcal{F}_b, \mathcal{M})$ is a *fuzzy Banach space*, hence the sequence converges to a point $Q_n(r_1) \in \mathcal{F}_b$. Define $Q_n(r_1) = \mathcal{V}_s \to \mathcal{F}_b$ by

$$Q_n(r_1) = \mathcal{M} - \lim_{b\to\infty} \frac{f_q(k^b r_1)}{k^{4b}}$$

all $r_1 \in \mathcal{V}_s$. Letting $a = 0$ in (3.11), we get

$$\mathcal{M}\Big(\frac{f_q(k^b r_1)}{k^{4b}} - f_q(r_1), c\Big) \geq \mathcal{M}'\Big(\eta(r_1, 0), \frac{c}{\sum_{j=0}^{b-1} \frac{p^j}{2k^{4(j+1)}}}\Big), \tag{3.12}$$

for all $r_1 \in \mathcal{V}_s$ and all $c > 0$. Taking the limit $b \to \infty$ and again using definition of *fuzzy normed space*, we get,

$$\mathcal{M}(f_q(r_1) - Q_n(r_1), c) \geq \mathcal{M}'\Big(\eta(r_1, 0), \frac{c}{2}(k^4 - p)\Big),$$

for all $r_1 \in \mathcal{V}_s$ and all $c > 0$. Next we claim that $Q_n$ is quartic. Replacing $r_1, r_2$ by $k^b r_1, k^b r_2$ in (3.2), we have,

$$\mathcal{M}\Big(\frac{1}{k^{4b}} \hat{D} f_q(k^b r_1, k^b r_2), c\Big) \geq \mathcal{M}'(\eta(k^b r_1, k^b r_2), k^{4b} c)$$

for all $r_1, r_2 \in \mathcal{V}_s$ and all $c > 0$. Since,

$$\lim_{b\to\infty} \mathcal{M}'(\eta(k^b r_1, k^b r_2), k^{4b} c) = 1,$$

$Q_n$ satisfies (2.1). Hence, $Q_n(r_1) = \mathcal{V}_s \to \mathcal{F}_b$ is quartic.
For uniqueness, let $Q_n'(r_1) = \mathcal{V}_s \to \mathcal{F}_b$ be another quartic mapping satisfying (3.3). For $r_1 \in \mathcal{V}_s$, we have $Q_n(k^b r_1) = k^{4b} Q_n(r_1)$ and $Q_n'(k^b r_1) = k^{4b} Q_n'(r_1)$ for all $b \in \mathbb{N}$. It follows from (3.3) that

$$\begin{aligned}
\mathcal{M}(Q_n(r_1) - Q_n'(r_1), c) &= \mathcal{M}\Big(\frac{Q_n(k^b r_1)}{k^{4b}} - \frac{Q_n'(k^b r_1)}{k^{4b}}, c\Big) \\
&\geq min\Big\{\mathcal{M}\Big(\frac{Q_n(k^b r_1)}{k^{4b}} - \frac{Q_n'(k^b r_1)}{k^{4b}}, \frac{c}{2}\Big), \mathcal{M}\Big(\frac{Q_n(k^b r_1)}{k^{4b}} - \frac{Q_n'(k^b r_1)}{k^{4b}}, \frac{c}{2}\Big)\Big\} \\
&\geq \mathcal{M}'\Big(\eta(k^b r_1, 0), \frac{k^{4b} c(k^4 - p)}{4}\Big) \\
&\geq \mathcal{M}'\Big(\eta(r_1, 0), \frac{k^{4b} c(k^4 - p)}{4p^b}\Big)
\end{aligned}$$

for all $r_1 \in \mathcal{V}_s$ and all $c > 0$. Since, $\lim_{b\to\infty} \frac{k^{4b} c(k^4 - p)}{4p^b} = \infty$, we obtain

$$\lim_{b\to\infty} \mathcal{M}'\Big(\eta(r_1, 0), \frac{k^{4b} c(k^4 - p)}{4p^b}\Big) = 1.$$

So, we conclude that $Q_n(r_1) = Q_n'(r_1)$ for all $r_1 \in \mathcal{V}_s$. Thus $Q_n : \mathcal{V}_s \to \mathcal{F}_b$ is unique quartic mapping as desired.
We can demonstrate the result in the same manner for $j = -1$. This completes the proof. □

## 3.2 Corollary

Let $f_q : \mathcal{V}_s \to \mathcal{F}_b$ be a mapping which maps zero to zero and satisfies

$$\mathcal{M}(\hat{D} f_q(r_1, r_2), c) \geq \begin{cases} \mathcal{M}'(\Lambda, c), & \\ \mathcal{M}'(\Lambda(||r_1||^\beta + ||r_2||^\beta), c), & \beta \neq 4 \\ \mathcal{M}'(\Lambda(||r_1||^\beta ||r_2||^\beta + ||r_1||^{2\beta} + ||r_2||^{2\beta}), c), & \beta \neq 2 \end{cases} \tag{3.13}$$

for all $r_1, r_2 \in \mathcal{V}_s$ and all $c > 0$, where $\Lambda$ and $\beta$ are real constants with $\Lambda > 0$. Then there exist a unique quartic mapping $Q_n : \mathcal{V}_s \to \mathcal{F}_b$ such that

$$\mathcal{M}(f_q(r_1) - Q_n(r_1), c) \geq \begin{cases} \mathcal{M}'(\Lambda, \frac{c|k^4-1|}{2}), & \\ \mathcal{M}'(\Lambda ||r_1||^\beta, \frac{c|k^4-k^\beta|}{2}), & \beta \neq 4 \\ \mathcal{M}'(\Lambda ||r_1||^{2\beta}, \frac{c|k^4-k^{2\beta}|}{2}), & \beta \neq 2 \end{cases} \tag{3.14}$$

for all $r_1 \in \mathcal{V}_s$ and all $c > 0$.

*Proof.* Taking

$$\eta(r_1, r_2) = \begin{cases} \Lambda, & \\ \Lambda(||r_1||^\beta + ||r_2||^\beta), & \beta \neq 4 \\ \Lambda(||r_1||^\beta ||r_2||^\beta + ||r_1||^{2\beta} + ||r_2||^{2\beta}), & \beta \neq 2 \end{cases}$$

for all $r_1, r_2 \in V$ in Theorem [2.1], we get the desired result [18]. □

# 4 Conclusion

In this paper a new type GQFE is introduced and its Generalized HUR stability is proved in fuzzy normed space. In future, stability of the equation can also be proved in various spaces like random normed space, modular space, non-Archimedean normed space, quasi- $\beta$-normed spaces etc.

# Competing Interests

Author has declared that no competing interests exist.

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