Isac's Cones

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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ABSTRACT

This is a very short research work representing an homage to the regretted Professor George Isac, Department of Mathematics and Computer Science, Royal Military College of Canada, P.O. 17000, Kingston, Ontario, Canada, K7K 7B4. Professor Isac introduced the notion of "nuclear cone" in 1981, published in 1983 and called later as "supernormal cone" since it appears stronger than the usual concept of "normal cone". For the first time, we named these convex cones as "Isac's Cones" in 2009 , after the acceptance on professor Isac's part. This study is devoted to Isac's cones, including significant examples, comments and several pertinent references, with the remark that this notion has its real place in Hausdorff locally convex spaces not in the normed linear spaces, having strong implications and applications in the efficiency and optimization. Isac's cones represent the largest class of convex cones discovered till now in separated locally convex spaces ensuring the existence and important properties for the efficient points under completeness instead of compactness.

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1. INTRODUCTION

This is a very short research work representing an homage to Professor George Isac, Department of Mathematics and Computer Science, Royal Military College of Canada, P.O. 17000, Kingston, Ontario, Canada, K7K 7B4. Professor Isac introduced the notion of "nuclear cone" in [1], published in [2] and called later on "supernormal cone" since it appears stronger than the usual concept of "normal cone". For the first time, we named these convex cones as "Isac'c Cones" in [3], after the acceptance on professor Isac's part. This study is devoted to Isac's cones, including significant examples, comments and several pertinent references, with the remark that this notion is more interesting in the Hausdorff locally convex spaces as in the normed linear spaces, having strong implications and applications in the efficiency and the optimization. A generalization of Isac's cones in the general Vector Spaces was given by us in [4].

2. ISAC'S (NUCLEAR OR SUPER-NORMAL) CONES

Throughout the research works devoted to nuclear (supernormal) cones professor Isac considered any locally convex space in the sense of the next definition.

Definition 1. [5]. *A locally convex space is any couple* $(X, Spec(X))$

which is composed of a real linear space X and a family $Spec(X)$ of seminoms on X

such that:

(i) $\chi p \in Spec(X), \forall \chi \in R_+, p \in Spec(X)$;

(ii) *if $p \in Spec(X)$ and q is an arbitrary seminorm on X such that $q \leq p$, then $q \in Spec(X)$;*

(iii) $\sup(p_1, p_2) \in Spec(X), \forall p_1, p_2 \in Spec(X)$ where
$\sup(p_1, p_2)(x) = \sup(p_1(x), p_2(x)), \forall x \in X$

It is well known [5] that whenever such a family as this $Spec(X)$ is given on a real vector space X, there exists a locally convex topology τ on X such that (X, τ) is a topological linear space and a seminorm p on X is τ - continous iff $p \in Spec(X)$. A a non-empty subset B of $Spec(X)$ is a *base* for it if for every $p \in Spec(X)$ there exist $\chi \rhd 0$ and $q \in$ B such that $p \leq \chi q$ and (X, τ) is a Hausdorff locally convex space iff $Spec(X)$ has a base B , named *Hausdorff base* , with the property that $\{x \in X : p(x) = 0, \forall p \in B\} = \{\theta\}$ where θ is the null vector in X . In this research paper we will suppose that the space (X, τ) sometimes denoted by X is a Hausdorff locally convex space. Every non-empty subset K of X satisfying the following properties: $K + K \subseteq K$ and $\chi K \subseteq K, \forall \chi \in R_+$ is named *convex cone.* If, in addition, $K \cap K = \{\theta\}$, then K is called *pointed.* Clearly, any pointed convex cone K in X generates an ordering on X defined by $x \leq y(x, y \in X)$ iff $y - x \in K$. If X^* is the dual of X , then the *dual cone* of K is defined by $K^* = \{x^* \in X^* : x^*(x) \geq 0, \forall x \in K\}$ and its corresponding *polar* is $K^0 = -K$. We recall that a pointed convex cone $K \subset (X, Spec(X))$ is *normal* with respect to the topology defined by $Spec(X)$ if it fulfils one of the next equivalent assertions:

(i) *There exists at a base Ω of neighborhoods for the origin θ in X such that $V = (V + K) \cap (V - K), \forall V \in \Omega$;*

(ii) *There exists a base B of $Spec(X)$ with $p(x) \leq p(y), \forall x, y \in K, x \leq y, \forall p \in B$;*

(iii) *For any two nets $\{x_i\}_{i \in I}, \{y_i\}_{i \in I} \subset K$ with $\theta \leq x_i \leq y_i, \forall i \in I$ and $\lim y_i = \theta$ it follows that $\lim x_i = \theta$. In particular, a convex cone K is normal in a normed linear space $(E, \| . \|)$ iff there exists $t \in (0, \infty)$ such that $x, y \in E$ and $y - x \in K$ implies that $\| x \| \leqslant t \| y \|$.*

It is well known that the concept of *normal cone* is the most important notion in the theory and applications of convex cones in topological ordered vector spaces. Thus, for example, *for*

every separated locally convex space $(X, Spec(X))$ and any closed normal cone $K \subset (X, Spec(X))$ we have $X^* = K^* - K^*$ (see, for instance, [6,7]). Each pointed convex cone $K \subset (X, Spec(X))$ for which there exists a non-empty, convex bounded set $T \subseteq X$ such that $0 \notin \overline{T}$ and $K = \bigcup_{\chi \geq 0} \chi T$ is called *well-based*. A cone $K \subset (X, Spec(X))$ *is well-based iff there exists a base* $B = \{p_i\}_{i \in I}$ *of* $Spec(X)$ *and a linear continuous functional* $f \in K^*$ *such that for every* $p_i \in B$ *there exists* $c_i > 0$ *with* $c_i p_i(x) \leq f(x), \forall x \in K$ ([6,7,1,2,8]). Clearly, every well-based cone is a normal cone, but, in general, the converse is not true, as we can see in the examples below, starting from the next basic notion.

Definition 2. ([1,2]). *In a Hausdorff locally convex space* $(X, Spec(X))$ *a pointed convex cone* $K \subset X$ *is nuclear (supernormal) with respect to the topolgy induced by* $Spec(X)$ *if there exists a base* $B = \{p_i\}_{i \in I}$ *of* $Spec(X)$ *such that for every* $p_i \in B$ *there exists* $f_i \in X^*$ *with* $p_i(x) \leq f_i(x), \forall x \in K$.

Remark 1. For the first time, we called any such as this cone *"Isac's cone"* in [4], taking into account that the above definition of locally convex spaces is equivalent with the following: Let X be a real or complex linear space and $P = \{p_\alpha : \alpha \in A\}$ a family of seminorms defined on X. For every $x \in X, \varepsilon > 0$ and $n \in N^*$ let

$$V(x; p_1, p_2, ..., p_n; \varepsilon) = \{y \in X : p_\alpha(y - x) < \varepsilon, \forall \alpha = \overline{1, n}\}, \text{ then the family}$$

$$\varsigma_0(x) = \{V(x; p_1, p_2, ..., p_n; \varepsilon) : n \in N^*, p_\alpha \in P, \alpha = \overline{1, n}, \varepsilon > 0\}\}$$

has the properties:

(V$_1$) $x \in V, \forall V \in \varsigma_0(x)$;

(V$_2$) $\forall V_1, V_2 \in \varsigma_0(x), \exists V_3 \in \varsigma_0(x) : V_3 \subseteq V_1 \cap V_2$;

(V$_3$) $\forall V \in \varsigma_0(x), \exists U \in \varsigma_0(x), U \subseteq V$ such that $\forall y \in U, \exists W \in \varsigma_0(y)$ with $W \subseteq V$.

Therefore, $\varsigma_0(x)$ is *a base of neighbourhoods* for x and taking $\varsigma(x) = \{V \subseteq X : \exists U \in \varsigma_0(x)$ with $U \subseteq V\}$, the set

$\tau = \{D \subseteq X : D \in \varsigma(x), \forall x \in D\} \cup \{\varnothing\}$ is the locally convex topology generated by the family P. Obviously, the usual operations which induce the structure of linear space on X are continuous with respect to this topology. The corresponding topological space (X, τ) is a Hausdorff locally convex space iff the family P is sufficient, that is, $\forall x_0 \in X \setminus \{\theta\}, \exists p_\alpha \in P$ with $p_\alpha(x_0) \neq 0$. In this context, a convex cone $K \subset X$ is an Isac's cone iff $\forall p_\alpha \in P, \exists f_\alpha \in X^* : p_\alpha(x) \leq f_\alpha(x), \forall x \in K$. The *best, special, refined and non - trivial Isac's cones classes associated to the sets of all normal cones in arbitrary Hausdorff locally convex spaces was introduced and studied in* [9] *as the full nuclear cones, these families of convex cones being defined as follows: if* $(X, Spec(X))$ *is an arbitrary locally* convex space, $B \subset Spec(X)$ is a Hausdorff base of $Spec(X)$ and $K \subset X$ is a normal cone, then for any mapping $\varphi : B \to K^* \setminus \{0\}$ one says that the set

$K_{\varphi} = \{x \in X : p(x) \le \varphi(p)(x), \forall p \in B\}$ is *a full nuclear cone* associated to K whenever $K_{\varphi} \ne \{\theta\}$. Taking into account that in any real normed linear space $(E, \|\cdot\|)$ a non-empty set $T \subset E$ is called *a Bishop-Phelps cone* if there exists y^* in the usual dual space E^* of E and $\alpha \in (0,1]$ such that $T = \{y \in E : \alpha\|y\| \le y^*(y)\}$ and the applications of such as these cones in the Nonlinear Analysis together with the Vectorial Optimization induced by the vector-valued mappings, we conclude that, for the Haudorff locally convex spaces, *the full nuclear cones* are *the best generalizations* of the *Bishop-Phelps cones* in the normed vector spaces.

In the next considerations we offer *significant examples* and *adequate remarks* on *the supernormal cones. The existence* of *the efficient points* and *important properties* of *the efficient points sets are ensured* in *the separated locally convex spaces ordered* by the (*weak*) *supernormal cones named by us "Isac's cones", through the agency of the* (*weak*) *completeness* instead of *the compactness (the reader is referred to* [7,1,2,8,10,11,12,13,9,14,15,16,17, 18,19,20,21,22,23,24,3,4,25,26] *and so on*).

Theorem 1. [27]. *A convex and normal cone* K *in a Hausdorff locally convex space is supernormal if and only if every net of* K *weakly convergent to zero converges to zero in the locally convex topology.*

Let us consider some pertinent examples

1. *Any convex, closed* and *pointed cone* in *an arbitrary* usual *Euclidean space* $R^k (k \in N^*)$ is *supernormal*.
2. In *every locally convex space* any *well-based convex cone is an Isac's cone*.
3. *A convex cone* is *an Isac's cone* in *a normed linear space if and only if* it is *well-based*.
4. Let $n \in N^*$ be *arbitrary fixed* and let Y be the space of *all real symmetric (n, n) matrices ordered* by *the pointed, convex cone* C = $\{A \in Y : x^T Ax \ge 0, \forall x \in R^n\}$. Then, Y is *a real Hilbert space with respect to the scalar product defined by <A, B> =trace (A. B) for all A, B∈ Y and C is well-based by B={A∈ C :<A, I>=1} where I denotes the identity matrix.*

5. *Every pointed, locally* or *weakly compact convex cone* in *any Hausdorff locally convex space is an Isac's cone.*
6. *A convex cone is an Isac's cone in any nuclear space* [28] *if and only if it is a normal cone.*
7. In *any Hausdorff locally convex space* a *convex cone* is *an weakly Isac's cone if and only if it is weakly normal.*
8. In $L^p([a, b])$, $(p \ge 1)$, *the convex cone* $K_p = \{x \in L^p([a, b]) : x(t) \ge 0$ *almost everywhere*$\}$ is *an Isac's cone if and only if p=1, being well based in this case* by *the set* $B = \{x \in K_1 : \int_a^b x(t)dt = 1\}$. Indeed, *if p>1, then the sequence* (x_n) *defined by*

$$x_n(t) = \begin{cases} n^{1/p}, & a \le t \le a+(b-a)/2n \\ 0, & a+(b-a)/2n < t \le b \end{cases} \quad n \in N$$

converges to 0 in the weak topology but not in its usual norm topology. Therefore, by virtue of the Theorem 1, K_p *is not an Isac's cone. Generally, for every p>1,* K_p *has a base* $B = \{x \in K_p : \int_a^b x(t)dt = 1\}$ *which is unbounded. Any convex cone generated by every closed and bounded set* $B_t = \{x \in B : \int_a^b |x(t)|^p dt \le t\}$ *with t≧0 is certainly an Isac's cone. A similar result holds for* $L^p(R)$. *Thus, if we consider a countable family* (A_n) *of disjoint sets which covers R such that* $\mu (A_n) = 1$ *for all n in N, where* μ *is the Lebesgue measure, then the sequence* (y_n) *given by* $y_n(t) = 1$ *if* $t \in A_n$ *and* $y_n(t) = 0$ *for* $t \in R \backslash A_n$ *converges weakly to zero while it is not convergent to zero in the norm topology. Taking into account the above Theorem 1, it follows that the usual positive cone in* $L^p(R)$ *is not an Isac's cone if p>1, that is, it is not well-based in all these cases.*

However, these cones are normal for every p $\ge$. The same conclusion concerning the non-supernormality is valid for the positive orthant of any usual Orlicz space.

9. In $l^p (p \ge 1)$ *equipped with the usual norm* $\|\cdot\|_p$ *the positive cone*

$C_p = \{(x_n) \in l^p : x_n \ge 0 \text{ for all } n \in N\}$ is also *normal* with respect to *its usual norm topology*, but *it is not an Isac's cone excepting the case p = 1.* Indeed, *for every p>1, the sequence* (e_n) *having*

1 at the nth coordinate and *zeros elsewhere converges to zero in the weak topology, but not in the norm topology* and by *virtue of Theorem 1* it follows that C_p *is not an Isac's cone*. For $p = 1$, C_p is *well-based* by *the set* B = $\{x \in C_1 : \|x\|_1 = 1\}$ and *Proposition 5 given in* [2] *ensures that* it is *an Isac's cone*. If we consider *in this case the locally convex topology in* l^1 *defined by the*

$$\text{seminorms } p_n((x_k)) = \sum_{k=0}^{n} |x_k| \text{ for every } (x_k) \text{ in } l^1$$

and $n \in N$, which is *weaker than its usual weak* topology, then *the usual positive cone remains an Isac's cone* with respect to *this topology* (now *it is normal in a nuclear space* and *one applies Proposition 6* of [2]), *but it is not well based*. Taking into account *the concept* of *H-locally convex space* introduced in [29] and defined as *any Hausdorff locally convex space* with *the seminorms satisfying the parallelogram law* and *the property* that *every nuclear space is also a H-locally convex space* with respect to *an equivalent system* of *seminorms* [28], *the above example shows* that, in *a H-locally convex space, a proper convex cone may be an Isac's cone without to be well-based*. Moreover, *if we consider in* l^2 *the H-locally convex topology induced by the seminorms*

$$\tilde{p}_n((x_k)) = \left(\sum_{i \geq n} |x_i|^2\right)^{1/2}, \; n \in N, \; (x_k) \in \ell^2,$$

then, the convex cone $C_2 = \{(x_k) \in l^2 : x_k \geq 0,$ for all $k \in N\}$ is normal in *the H-locally convex space* $(l^2, \{\tilde{p}_n\}_{n \in N})$, *but it is not a supernormal cone* because *the same sequence* (e_k) *is weakly convergent to zero* while $(\overline{p}_n(e_k))$ *is convergent to 1 for each* $n \in N$ and *one applies again the Theorem1*. Another *interesting example* of *normal cone* in *a H-locally convex space* which *is not supernormal is the usual positive cone* in *the space* $L^2_{loc}(R)$ *of all functions from R to C which are square integrable over any nontrivial, finite interval of R*, endowed with *the system of the seminorms*

$\{\overline{p}_n : n \in N\}$ defined by $\overline{p}_n(x) = \left(\int_{-n}^{n} |x(t)|^2 dt\right)^{1/2}$ *for*

every x in $L^2_{loc}(R)$. In *this case, the sequence* (x_k) given by:

$$x_k(t) = \begin{cases} 0, & t \in (-\infty, 0) \cup (1/k, +\infty) \\ \sqrt{k}, & t \in [0, 1/k] \end{cases}$$

converges weakly to zero, but *it is not convergent to zero in the H-locally convex topology. The results follows* by *Theorem 1*. It is clear that *every weak topology is a H-locally convex topology* and, *in all these cases, the supernormality of the convex cones coincides* with *the normality* thanks *to the Corollary of the Proposition 2 given in* [2].

10. In *the space C([a, b]) of all continuous, real valued functions defined on every non-trivial, compact interval [a, b] equipped with the usual supremum norm the convex cone K = {x∈C([a, b]): x is concave, x(a)=x(b) = 0 and x(t)≥0 for all t∈[a, b]}* is *supernormal, being well based by the set {x∈K: x(t_0) =1} for some arbitrary* $t_0 \in [a,b]$. *The hypothesis that all x∈K are concave is essential for the supernormality.*

11. *The convex cone* of *all nonnegative sequences* in *the space* of *all absolutely convergent sequences* is *the dual* of *the usual positive cone* in *the space* of *all convergent sequences*. Consequently, *it has a weak star compact base* and, hence, *it is a weak star supernormal cone.*

12. In l^∞ or in c_0 equipped with the supremum norm, the convex cone consisting of all sequences having all partial sums non-negative is not normal, so it is not supernormal.

13. In *every Hausdorff locally convex space any normal cone is supernormal* with respect to *the weak topology*.

14. In *every locally convex lattice which is a (L)-space the ordering cone is supernormal* (see also *the Example 7* given in [11]).

15. If we consider *the space of all locally integrable functions* on *a locally compact space Y* with respect to *a Radon measure* μ endowed with *the topology induced by the family of seminorms* $\{p_A\}$ *where* $p_A(f) = \int_A |f(x)| d\mu$ *for every non-empty* and *compact subset A* of *Y* and *every locally integrable function f*, then *the convex cone K = {f: f(x)≥0, x∈Y} is supernormal.*

16. If *Z is any locally convex lattice ordered* by *an arbitrary convex cone K* and Z^* is its *topological dual ordered* by *the corresponding dual cone* K^*, then *the cone K is supernormal* with respect to *the locally convex topology defined on Z* by *the neighbourhood base at the origin* $\{[-f, f]^\circ\}_{f \in K^*}$.

17. In *every regular vector space (E, K)* (that is, *the order dual* E^* *separates the points of*

E) with *the property* that *E = K - K the convex cone K is an Isac's cone* with respect to *the topology* defined in *the preceding example.*

18. Any *semicomplete cone* in *a Hausdorff locally convex space* is *supernormal* (for this concept see *the Example 11* of [11]).

Remark 2. Clearly, *if a convex cone K is supernormal in a normed space*, then *K admits a strictly positive, linear and continuous functional*, that is, *there exists a linear, continuous functional f* such that *f(k)>0 for all* $k \in K \setminus \{0\}$. *Generally, the converse is not true, even in a Banach space, as we can see in the following examples.*

19. *If one considers in the usual space* l^p $\left(1 \le p \le \infty\right)$ *the usual ordering convex cone*

    $K = \ell_+^p = \left\{ x = (x_i) \in \ell^p : x_i \ge 0 \text{ for every } i \in N \right\}$ *of all the infinite vectors with non-negative components, then the functional φ defined by* $\varphi(k) = \sum_{i=1}^{\infty} k_i$ *for any* $k = (k_i) \in l^p$ *is linear, continuous and strictly positive. But, as we have seen in the above considerations (Example 9), this cone is supernormal if and only if p = 1.*

20. *Let K be the usual positive cone* $L^p_+ = \{x \in L^p([a, b]) : x(t) \ge 0$ *almost everywhere}* in $L^p([a, b])$ $(1 \le p \le \infty)$. *Then, the linear and continuous functional ψ on* $L^p([a, b])$ *given by* $\psi(x) = \int_a^b x(t) dt$ *for every* $x \in L^p([a, b])$ *is strictly positive on K while K is supernormal (see the above Example 8) if and only if p = 1. The same conclusion is valid for the space* $L^p(R)$ $(1 \le p \le \infty)$. *Therefore,* l^1_+ *and* L^1_+ *are Isac's cones, with empty topological interiors, and for every* $p \in (1, +\infty)$ *it follows that* l^p_+ *and* L^p_+ *are normal cones with empty interiors, which are not supernormal. Hence, these convex cones are not well based. Very simple examples of Isac's cones having non-empty topological interiors are* R^n_+ $(n \in N^*)$.

Remark 3. In *the order complete vector lattice B([a, b])* of *all bounded, real valued functions on a compact non-singleton interval [a, b]* endowed with *its usual norm the standard positive cone* $K = \{u \in B([a, b]) : u(t) \ge 0 \text{ for all } t \in ([a, b])\}$ *is normal but it has not a base*, that is, *it is not supernormal. However, this cone has non-empty interior. If we consider the linear space* l^1 *endowed with the separated locally convex topology generated by the family* $\{p_n : n \in N\}$ *of seminorms defined by* $p_n(x) = \sum_{k=0}^n |x_k|$ *for every* $x = (x_k) \in l^1$, *then the convex cone* $K = \{x = (x_k) \in l^1 : x_k \ge 0 \text{ whenever } k \in N\}$ *is supernormal,* but *it is not well based.*

Remark 4. *The natural context of supernormality* (nuclearity) *for convex cones* is *any separated locally convex space. Isac, G. introduced the concept of "nuclear cone" in 1981, published it in 1983 and he showed that in a normed space a convex cone is nuclear* if and only if it is *well based* or equivalently *iff it is "with plastering",* the last concept being defined by *Krasnoselski, M. A.* in *fifties (see, for example, Krasnoselski, M. A., 1964 and so on) [30]. Such a convex cone was* initially called *"nuclear cone"* by *Isac, G. (1981)* because *in every nuclear space (Pietch, A., 1972) any normal cone is a nuclear cone in Isac's sense (Proposition 6 of Isac, G., 1983).* Afterwards, *since the nuclear cone introduced by Isac appears as a reinforcement of the normal cone, it was called supernormal. The class of supernormal cones in Hausdorff locally convex spaces was initially imposed by the theory and the applications of the efficient (Pareto minimum type) points (especially existence conditions based on completeness instead of compactness were decisive together with the main properties of the efficient points sets), the study of critical points for dynamical systems and conical support points and their importance was very well illustrated by important results, examples and comments in the specified references and in other connected papers. It is also very significant to mention again* that *the concept of supernormality introduced by Isac, G. (1981) is not a simple generalization of the corresponding notion defined in normed linear spaces by Krasnoselski, M. A. and his colleagues in the fifties. Thus, for example, Isac's supernormality attached to the convex cones has his sense in every Hausdorff locally convex space identically with the well known Grothendieck's nuclearity. By analogy* with *the fact* that *a normed space is nuclear in Grothendieck's sense if and only if it is isomorphe with an usual Euclidean space, a convex cone is supernormal in a normed space if and only if it is well based, that is, it is generated by a convex bounded set which does not contain the origin in its closure. Beside Pareto type*

optimization, we also mention Isac's significant contributions, through the agency of supernormal cones, to the convex cones in product linear spaces and Ekeland's variational type principles (Isac.G., 2003; Isac, G., Tammer, Chr., 2003). Therefore*, the more appropriate background for Isac's cones is any separated locally convex space.*

3. CONCLUSIONS AND FUTURE RESEARCH DIRECTIONS

The family of Isac's cones *represents the largest class of ordering cones in Hausdorff locally convex spaces ensuring the existence and the adequate properties for the efficient points sets involved in the general optimization, following different completeness types instead of compactness.* Consequently, *one of the main goal of the next research is to identify new applications of Isac's cones in the efficiency projected in the best approximation problems, the set - valuet fixed point theory including the dynamical systems and the nuclearity of the linear vector spaces.*

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COMPETING INTERESTS

Author has declared that no competing interests exist.

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