



## **Buoyancy-Induced MHD Stagnation Point Flow of Williamson Fluid with Thermal Radiation**

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### **Authors' contributions**

*This work was carried out in collaboration among all authors. All authors jointly designed the study and wrote the first draft of the manuscript. All authors did the simulations independently and their results agreed. All authors read and approved the final manuscript.*

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## **ABSTRACT**

Flow of fluids subjected to thermal radiation has enormous application in polymer processing, glass blowing, cooling of nuclear reactant and harvesting solar energy. This paper considers the MHD stagnation point flow of non-Newtonian pseudoplastic Williamson fluid induced by buoyancy in the presence of thermal radiation. A system of nonlinear partial differential equations suitable to describe the MHD stagnation point flow of Williamson fluid over a stretching sheet is formulated and then transformed using similarity variables to boundary value ordinary differential equations. The graphs depicting the effect of thermal radiation parameter, buoyancy and electromagnetic force on the fluid velocity and temperature of the stagnation point flow are given and the results revealed that increase in buoyancy leads to an increase in the overall velocity of the flow but a decrease in the temperature of the flow.

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## 1 INTRODUCTION

Magnetohydrodynamics (MHD) revolves around the dynamism of fluids that transmit electricity when passed through a magnetic field [1]. Krishnamurthy et al. [2] listed these fluids as liquid metals (molten iron, mercury and gallium), water, ionized gases like the solar atmosphere and salt. Such flow has been found to be common in real life application such as liquid coating through photographic films and the extrusion from behind the polymeric sheet from expiring. The magnetic field induces an electric current through the conductive fluid that is in motion and this process is called electromagnetism. The induced current compels the liquid and also brings changes to the magnetic field. Rajendar and Babu [3] pointed out that every unit of the the fluid volume experiences MHD force and Malik and Salahuddin [4] identified the force as Lorentz force. By introducing the Maxwell equations of electromagnetism into the Navier Stokes equations, we have the equations governing MHD flow.

Brimmo and Qasaimeh [5] defined stagnation point flow as the scenario where a body of fluid lacks mobility through a given region such that the region has a zero local velocity. Stagnation-point flow, describing the fluid motion over a continuously stretching surface in presence of electromagnetic fields are significant in many engineering processes with applications in industries such as the metallurgy, polymer processing, glass blowing, glass-fibre production, paper production, plastic films drawing, filaments drawn through a quiescent electrically conducting fluid subject to a magnetic field and the purification of molten metal from non-metallic inclusions. The quality of the final products depend to a great extent on the rate of cooling at the stretching surface, thus for superior products the heat transfer should be controlled.

In several physical situations, thermal energy of a fluid is converted to electromagnetic energy

which is emitted from the fluid. This emission is called thermal radiation and its features are governed by its temperature. Some common examples of thermal radiation include the infrared radiation emitted by common household type of radiator (such as the electric heater), infrared emitted from some animals, infrared radiation from hot metal, etc. There is a tendency for a fluid to rise once an external source of heat is applied, this tendency is referred to as buoyancy. This happens a lot in industries where cooling and heating processes are of high importance.

Koo and Kleinstreuer [6] investigated stagnation point flow with viscous dissipation and established that the velocity and the transfer of heat are ever high. However, the temperature and skin fraction are ever low in the case of the stretching based paramy. Ishak et al. [7] analysed a stagnation point flow over stretching vertical sheet and reported that both the skin friction coefficient and the heat transfer rate increase with buoyancy.

In an investigation of an MHD stagnation point flow in the presence of chemical reaction, [8] established that magnetic parameter tends to promote the heat and mass transfer; consequently leading to the reduction of friction factor despite the presence of the inductive magnetic field. In a further research, [9] examine Williamson fluid flow over a stretching sheet and recorded that increase in Williamson parameter leads to decrease in both primary velocity and skim friction coefficient. Malik and Salahuddin [4] established that smaller Prandtl value (those  $Pr$  value that are not exceeding one) have low variation in viscous flow through non linear stretching sheet with the impact of viscous dissipation. Krishnamurthy [2] found out that magnetic parameter causes a decrease in the velocity and also leads to an increase in its temperature when the Magnetohydrodynamics flow of Williamson nanofluid is considered over non-linear surface. Sandeep et al. [10] found that magnetic field parameter have tendency to enhance the heat and mass transfer rate and

reduce the friction factor in presence of induced magnetic field. Bhatti et al. [11] identified that the energy got following the thermal radiation causes the raising of the temperature and thus leading to the increment in the energy expenditure being far from the surface of the plate. Kumar et al.[12], Khan et al.[13], Ch.Vittal et al. [14], Rajendarand Babu [3], and Hamid et al. [15] remarked in support of [2] that increase in magnetic parameter decreases the velocity profile while increasing temperature profile. Narender et al. [16] studied the effect of thermal radiation on the stagnation point flow of Williamson fluid over a stretching. Based on his findings, he established the Nusselt number falls and the coefficient of skin friction experiences increment like in the case of the Newtonian Williamson fluid aspect. Ahmed et al. [17] studied thermal radiation effect on MHD stagnation point flow of Williamson fluid flow over stretching sheet and it was found that the value of Nusselt number decreases as skin friction coefficient increases with an increase in non-Newtonian Williamson fluid parameter velocity. Narender et al. [16] examine MHD stagnation point of nanofluid over radially stretching sheet and it was found that magnetic parameter decelerate the velocity where as an opposite trend has been observed for the temperature and concentration magnetic fields by considering the Casson fluid, for Casson fluid the higher estimation of the Casson fluid parameter escalates the velocity, temperature. Megaheed [18] considered Williamson fluid flow due to a nonlinear stretching sheet with viscous dissipation and thermal in the study viscosity and fluid conductivity are assume to vary with temperature and the noted that thermal radiation parameter and the Eckert number has an effect on the temperature distribution and thicken thermal region which leads to an increase in the local Nusselt number and decrease in local skin-friction coefficient while rise temperature is cause by an increase in both Williamson viscosity parameter. Panezai et al.[19] examined the influence of thermal radiation on two-dimensional incompressible magneto-hydrodynamic mixed convective heat transfer flow of Williamson fluid flowing past a porous wedge on a non linear stretching surface and the free stream experiencing external forces on the fluid and it was concluded that non

-dimensional velocity profile increases with an increase in wedge angle parameter while non-dimensional temperature profile decrease with an increase in the wedge angle parameter ,non-dimensional velocity deceases with an increase in magnetic parameter. Hashim et al. [20] investigated the effect of the thermal radiation on MHD stagnation point flow of Williamson fluid flow over a stretching surface; the flow is such that the wall experience linear stretching while the stream experiences external velocity and it was found that an increase in the Williamson parameter leads to a decrease in Nusselt number but an increase in skim friction coefficient. Other recent researches on MHD include [21, 22, 23].

From the ongoing, it is clear that a number of studies have been carried out to study the effects of various parameters on flow of Williamson fluids over linear and nonlinear stretching sheets but none has considered the combined effects of buoyancy and thermal radiation on MHD stagnation point flow of Williamson fluid over a linearly stretching sheet. In this study, we modify the work of Hashim et al. [20] to investigate the effects of Lorentz force, buoyancy force and thermal radiation on MHD stagnation point flow of Williamson fluid over a linearly stretching sheet.

## 2 GOVERNING EQUATIONS

For an incompressible MHD flow, the divergence free condition is imposed on the velocity  $\mathbf{V}$  so that the continuity equation

$$\nabla \cdot \mathbf{V} = 0$$

and the momentum equation is

$$\frac{D\mathbf{V}}{Dt} = \nabla \cdot \boldsymbol{\tau} + g\beta(T - T_\infty) + \mathbf{J} \times \mathbf{B}$$

where  $\boldsymbol{\tau}$  is the Cauchy stress tensor,  $D/Dt$  the material derivative,  $\mathbf{J}$  the current density,  $\mathbf{B} = \mathbf{B}_0 + \mathbf{b}$  is the total magnetic field which is the sum of the applied magnetic field  $\mathbf{B}_0$  and the induced magnetic field  $\mathbf{b}$ . The Cauchy stress tensor  $\boldsymbol{\tau}$  in a Williamson is

$$\boldsymbol{\tau} = -p\mathbf{I} + [\mu_\infty + (\mu_0 - \mu_\infty)(1 - \Gamma\gamma)^{-1}] \mathbf{A}_1 \quad (2.1)$$

where  $\mathbf{A}_1 = \nabla \mathbf{V} + (\nabla \mathbf{V})^T$  is the first Rivlin Erickson tensor,  $p$  is pressure,  $\mathbf{I}$  the identity tensor,  $\mu_\infty$  the infinite shear rate viscosity,  $\mu_0$  the zero shear rate viscosity,  $\Gamma$  the time constant and  $\gamma$  is defined as

$$\gamma = \sqrt{\frac{1}{2} \sum_i \sum_j \gamma_{ij} \gamma_{ji}} = \sqrt{\frac{1}{2} \Pi}$$

where  $\Pi = \text{trace}(\mathbf{A}_1^2)$ . In this study, take  $\mu_\infty = 0$  and  $\Gamma\gamma < 1$  and thus equation (2.1) reduces to

$$\tau = -p\mathbf{I} + \mu_0(1 + \Gamma\gamma)\mathbf{A}_1.$$

The components of the extra stress tensor are

$$\begin{aligned} \tau_{xx} &= 2\mu_0(1 + \Gamma\gamma) \frac{\partial u}{\partial x}, \quad \tau_{yy} = 2\mu_0(1 + \Gamma\gamma) \frac{\partial v}{\partial y} \\ \tau_{xy} &= \tau_{yx} = \mu_0(1 + \Gamma\gamma) \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right), \end{aligned}$$

and  $\tau_{xz} = \tau_{yz} = \tau_{zx} = \tau_{zy} = \tau_{zz} = 0$  and  $\mathbf{J} \times \mathbf{B} = (-\sigma B_0^2 u, 0, 0)$ . Consider a steady two dimensional flow of an incompressible Williamson fluid over a wall coinciding with the the plane  $y = 0$ . Two equal and opposing forces are applied along the  $x$ -axis to produce stretching, while keeping the origin fixed. Adopting the approach of [24], the continuity equation, momentum equations and the energy equation of such flow are;

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (2.2)$$

$$\begin{aligned} u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} &= \bar{U} \frac{\partial^2 u}{\partial y^2} + \sqrt{2}\bar{U}\Gamma \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial y^2} \\ &\quad + g\beta(T - T_\infty) - \sigma \frac{B_0^2 u}{\rho}, \end{aligned} \quad (2.3)$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} + \frac{1}{\rho c_P} \left( \frac{\partial q_r}{\partial y} \right). \quad (2.4)$$

where  $\bar{U} = \frac{\mu}{\rho}$ ,  $\alpha = \frac{K}{\rho c_P}$ , subject to the boundary conditions [25]

$$\text{at } y = 0 : \quad u = u_w(x) = ax, \quad v = 0, \quad T = T_w \quad (2.5)$$

$$\text{as } y \rightarrow \infty : \quad u \rightarrow u_\infty = bx, \quad T \rightarrow T_\infty. \quad (2.6)$$

Using the similarity variable

$$\eta = y\sqrt{\frac{a}{\bar{U}}}; \quad \varphi = \sqrt{a\bar{U}}xf(\eta), \quad \theta = \frac{T - T_\infty}{T_w - T_\infty}$$

The dimensionless equations are

$$f'''' + \lambda f'' f''' + Gr\theta - Mf' + ff'' - (f')^2 = 0, \quad (2.7)$$

$$\left(1 - \frac{4}{3}R\right)\theta'' + Prf\theta' = 0, \quad (2.8)$$

with the boundary conditions

$$\text{at } \eta = 0; \quad f = 0, \quad f' = 1, \quad \theta' = -Bi(1 - \theta), \quad (2.9)$$

$$\text{as } \eta \rightarrow \infty; \quad f' = \frac{b}{a}, \quad \theta \rightarrow 0, \quad (2.10)$$

where

$$\lambda = \Gamma \left( \frac{2a^3 x^2}{\bar{U}} \right)^{\frac{1}{2}}, \quad Gr = \frac{g\beta(T_w - T_\infty)}{a^2 x}, \quad (2.11)$$

$$M = -\frac{\sigma B_0^2}{a\rho}, \quad R = \frac{4\sigma T_\infty^3}{\alpha K \rho c_P}, \quad Pr = \frac{\bar{U}}{\alpha}, \quad (2.12)$$

$$Bi = \frac{h}{k} \sqrt{\frac{\bar{U}}{a}}. \quad (2.13)$$

the Williamson fluid parameter, Grashof number, magnetic field parameter, thermal radiation parameter, Prandtl number and Biot number respectively. The coefficient of skin friction is

$$\sqrt{2Re}C_f = \left( f''(0) + \frac{\lambda}{2} (f''(0))^2 \right). \quad (2.14)$$

### 3 DISCUSSION OF RESULTS

A buoyancy-induced steady flow of Williamson fluid over a stretching sheet in the presence of thermal radiation governed by the equations (2.2-2.4) is analysed in this section. The analytical solutions are difficult to obtain [26] and hence, the dimensionless equations (2.7-2.8) are numerically solved.

The Grashof number represents the presence of buoyancy in the flow. Buoyancy effect on the flow enhances the fluid motion, thereby increasing the both secondary and primary velocities (as shown in fig. 1. and fig. 2.) and since the system is losing energy to enhance the fluid motion, the temperature reduces (this is revealed in fig. 3.). Hence increase in buoyancy leads to an increase in the overall velocity of the flow but a decrease in

the temperature of the flow. As earlier remarked in the introduction, these results supports the findings of [16, 19, 20].

The presence of the magnetic field induces the electromagnetic force called the Lorentz force and hence increase in magnetic field parameter implies an increase in Lorentz force. It is well-known fact that the presence of Lorentz force retards the fluid motion and this is also supported by fig. 4. and fig. 5., where increase in Lorentz force leads to a decrease in both the primary and the secondary velocities of the flow. The Lorentz force generates more heat energy in the fluid and thus, raising the flow temperature; this is illustrated in fig. 6.

Increase in thermal radiation increases the heat generated to the surrounding by the system. This leads to a reduction in both the velocity and temperature of the flow. Fig. 7., Fig. 8. and Fig. 9. shows that the secondary velocity, the primary velocity and the temperature profiles decrease as thermal radiation parameter  $R$  increases.

Table 1. shows the variation of coefficient of skin friction with the Grashof number, magnetic field parameter and thermal radiation parameter. It is clear that the coefficient of skin friction increases with increasing Grashof number but decreases with magnetic field parameter and thermal radiation parameters

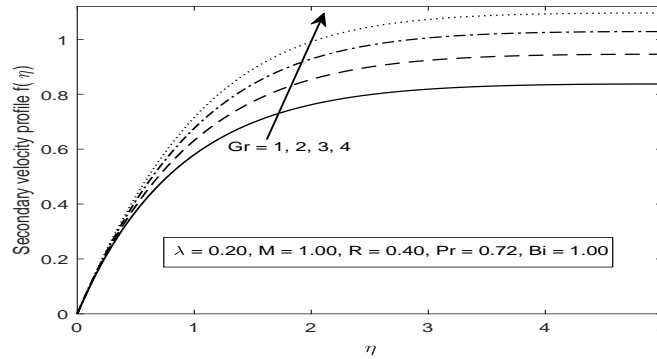


Fig. 1. Variation of secondary velocity with increasing buoyancy

Table 1. Variation of coefficient of skin friction with pertinent parameters with  $Pr = 0.72, Bi = 1, \lambda = 0.2$

Gr	M	R	$f''(0) + \frac{\lambda}{2} (f''(0))^2$
1.00			-1.106919024
2.00			-0.901599600
3.00			-0.710033136
4.00			-0.527840719
	1.00		-1.106919024
	3.00		-1.641978736
	5.00		-2.025718716
	7.00		-2.322204444
		0.20	-1.052806236
		0.40	-1.106919024
		0.60	-1.199387904
		0.70	-1.276329639

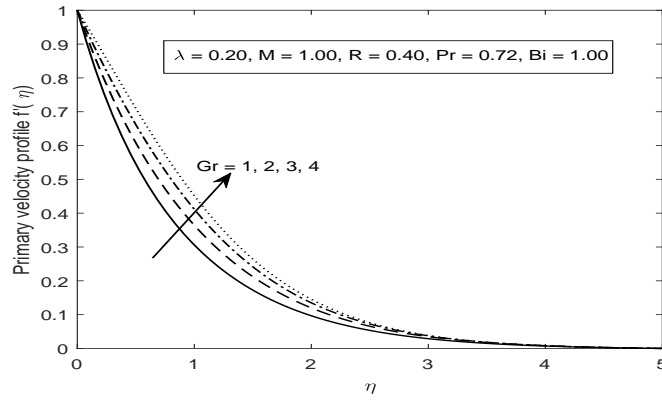


Fig. 2. Variation of primary velocity with increasing buoyancy

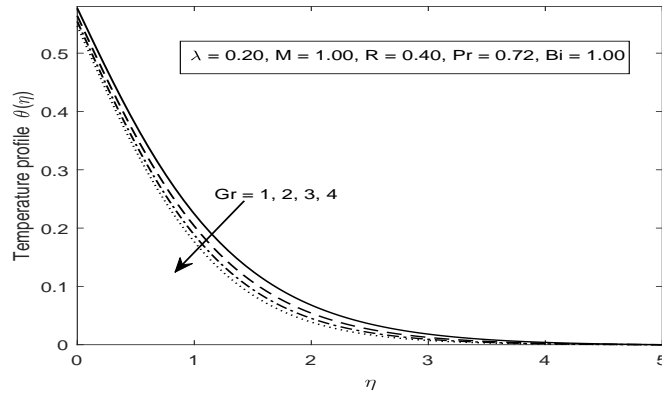


Fig. 3. Variation of temperature with increasing buoyancy

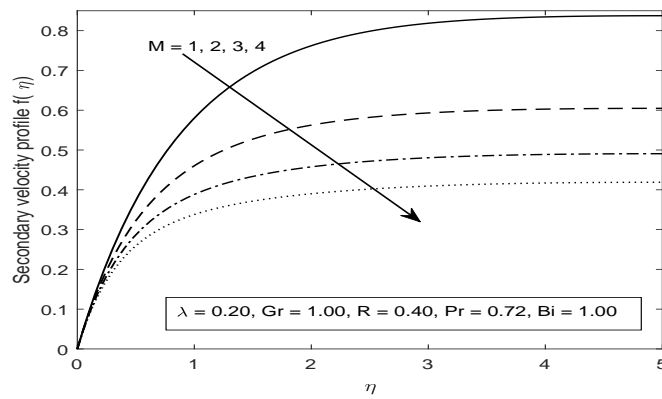


Fig. 4. Variation of secondary velocity with Lorentz force

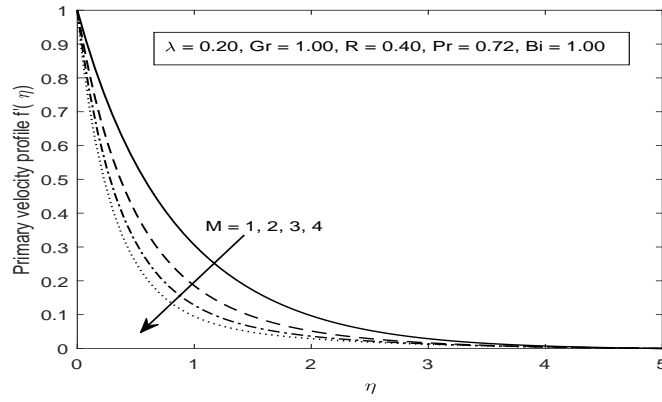


Fig. 5. Variation of primary velocity with increasing Lorentz force

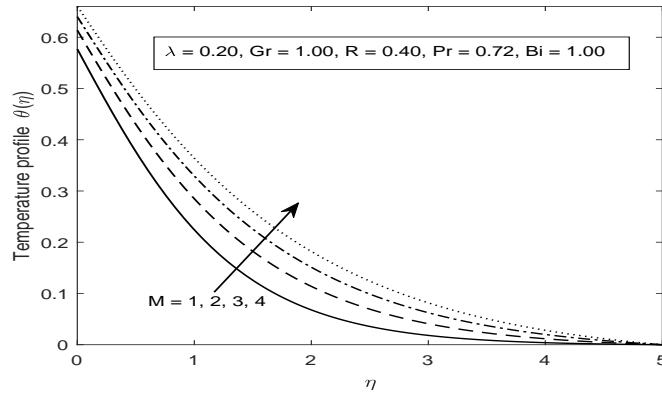


Fig. 6. Variation of temperature with increasing Lorentz force

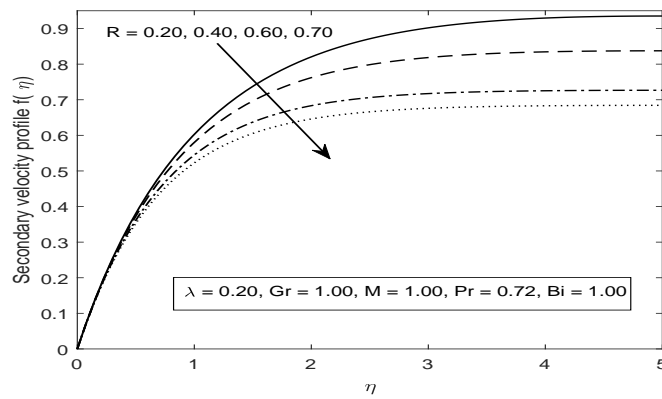
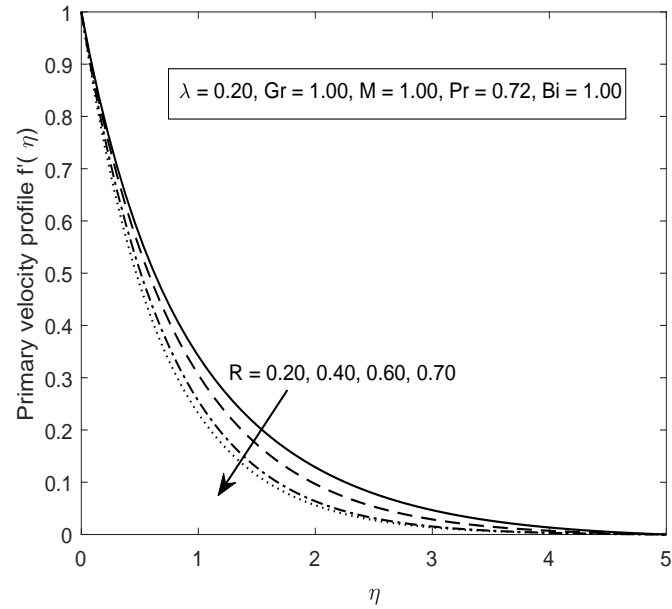
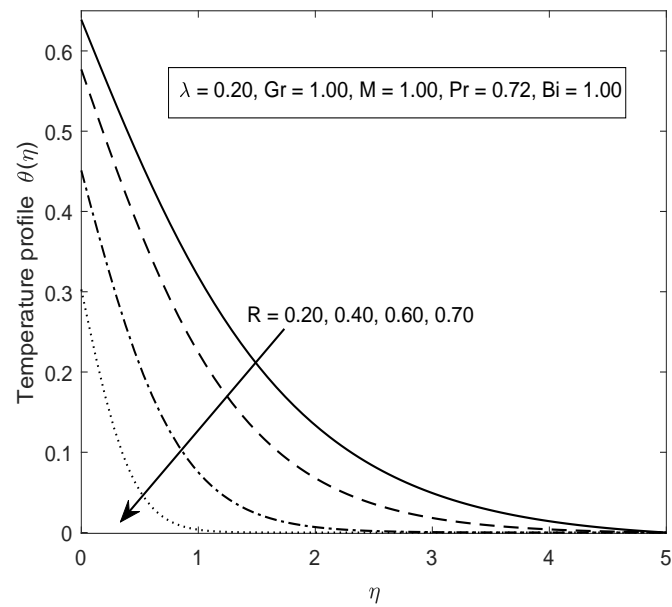


Fig. 7. Variation of secondary velocity with increasing buoyancy



**Fig. 8. Variation of primary velocity with increasing buoyancy**



**Fig. 9. Variation of temperature with increasing buoyancy**



## 4 CONCLUSION

A buoyancy-induced steady flow of Williamson fluid over a stretching sheet in the presence of thermal radiation governed is analysed in this paper. The outcome of the the analysis reveal that

1. Increase in buoyancy leads to an increase in the overall velocity of the flow but a decrease in the temperature of the flow.
2. Increase in Lorentz force leads to a decrease in the overall velocity of the flow but an increase in temperature.
3. Increase in thermal radiation leads to a reduction in both the velocity and temperature of the flow.
4. The coefficient of skin friction increases with increasing Grashof number but decreases with magnetic field parameter and thermal radiation parameters.

## COMPETING INTERESTS

Authors have declared that no competing interests exist.

## REFERENCES

- [1] Alven H. Nature. Existence of electromagnetic-hydrodynamic waves. 1942;(150):405406.
- [2] Krishnamurthy MR, Prasannakumara BC, Gireesha BJ, Gorla RSR. Effect of chemical reaction on mhd boundary layer flow and melting heat transfer of Williamson nanofluid in porous medium. Engineering Science and Technology, an International Journal. 2016;19(1):5361.
- [3] Rajendar P, Babu LA. Mhd stagnation point flow of Williamson nanofluid over an exponentially inclined stretching surface with thermal radiation and viscous dissipation. Journal of Nanofluids. 2018;7(4):683693.
- [4] Malik MY, Salahuddin T. Numerical solution of MHD stagnation point flow of williamson fluid model over a stretching cylinder. International Journal of Nonlinear Sciences and Numerical Simulation. 2015;16(3-4):161164.
- [5] Brimmo AT, Qasaimeh MA. Stagnation point flows in analytical chemistry and life sciences. RSC Advances. 2017;7(81):5120651232.
- [6] Koo J, Kleinstreuer C. Viscous dissipation effects in microtubes and microchannels. International Journal of Heat and Mass Transfer. 2004;47(14-16):31593169.
- [7] Ishak A, Nazar R, Pop I. Mixed convection boundary layers in the stagnation-point flow toward a stretching vertical sheet. Meccanica. 2006;41(5):509518.
- [8] Nadeem S, Hussain A, Malik MY, Hayat T. Series solutions for the stagnation flow of a second-grade fluid over a shrinking sheet. Applied Mathematics and Mechanics. 2009;30(10):1255.
- [9] Nadeem S, Hussain ST, Lee C. Flow of a williamson fluid over a stretching sheet. Brazilian Journal of Chemical Engineering. 2013;30(3):619625.
- [10] Sandeep N, Sulochana C, Animasaun IL. Stagnation-point flow of a Jeffrey nanofluid over a stretching surface with induced magnetic field and chemical reaction. In International Journal of Engineering Research in Africa: Trans Tech Publications. 2016;20:93-111.
- [11] Bhatti MM, Rashidi MM. Effects of thermo-diffusion and thermal radiation on williamson nanaofluid over a porous shrinking/shrinking sheet. Journal of Molecular Liquids. 2016;(221).
- [12] Kumar KG, Rudraswamy NG, Gireesha BJ, Manjunatha S. Nonlinear thermal radiation effect on williamson fluid with particle-liquid suspension past a stretching surface. Results in Physics. 2017;7:31963202.
- [13] Khan M, Malik MY, Salahuddin T, Rehman KU, Naseer M. Mhd flow of williamson nanofluid over a cone and plate with chemically reactive species. Journal of Molecular Liquids. 2017;231:580588.
- [14] Ch M.Vittal, Chenna Krishna Reddy, Vijayalaxmi T. Mhd stagnation point flow

- and heat transfer of williamson fluid over exponential stretching sheet embedded in a thermally stratified medium. *Global Journal of Pure and Applied Mathematics*. 2017;13(6):20332056.
- [15] Hamid A, Khan M, Hafeez A. Unsteady stagnation-point flow of williamson fluid generated by stretching/shrinking sheet with ohmic heating. *International Journal of Heat and Mass Transfer*. 2018;126:933940.
- [16] Narender G, Govardhan K, Sarma GS. Mhd stagnation point casson nanofluid flow over a radially stretching sheet. *Beilstein Archives*. 2019;2019(1):137.
- [17] Sameh E. Ahmed, Ramadan A. Mohamed, Abd Elraheem M. Aly, Mahmoud S. Soliman. Magneto hydrodynamic maxwell nanofluids flow over a stretching surface through a porous medium: Effects of non-linear thermal radiation, convective boundary conditions and heat generation/absorption. *World Academy of Science, Engineering and Technology International Journal of Aerospace and MEchanical Engineering*. , 2019;13(6):436443.
- [18] Ahmed M. Megaheed. Williamson fluid flow due to a nonlinearly stretching sheet with viscous dissipation and thermal radiation. *Journal of Egyptian Mathematical Society*. 2019;27(12).
- [19] Amina Panezai, Abdul Rehman, Naveed Sheikh, Saleem Iqbal, Israr Ahmed, Manzoor Iqbal, Muhammad Zulfiqar. Mixed convective magneto hydrodynamic heat transfer flow of Williamson fluid over a porous wedge. *American Journal of Mathematical and Computer Modelling*. 2019;4(3):6673.
- [20] Hasmawani Hashim, Muhammad Khairul Anuar Mohamed, Nazila Ishak, Norhafizah Md Sarif, Mohd Zuki Salleh. Thermal radiation effect on mhd stagnation point flow of williamson fluid over a stretching surface. *Journal of Physics: Conference Series*. 2019;1366(2019).
- [21] Veera Krishna M, Ali J. Chamkha. Hall and ion slip effects on MHD rotating boundary layer flow of nanofluid past an infinite vertical plate embedded in a porous medium. *Results in Physics*, 15:102652, 2019.
- [22] M. Veera Krishna and Ali J. Chamkha. Hall and ion slip effects on mhd rotating flow of elastico-viscous fluid through porous medium. *International Communications in Heat and Mass Transfer*. 2020;113:104494.
- [23] Veera Krishna M. Hall and ion slip effects on mhd laminar flow of an elastico-viscous (walters-b) fluid. *Heat Transfer*, n/a(n/a). 2020.
- [24] Anantha Kumar K, Sugunamma V, Sandeep N, Ramana Reddy JV. MHD stagnation point flow of williamson and casson fluids past an extended cylinder: A new heat flux model. *SN Applied Sciences*. 2019;1(7).
- [25] Winfred Nduku Mutuku. Ethylene glycol (eg)-based nanofluids as a coolant for automotive radiator. *Asia Pacific Journal on Computational Engineering*. 2016;3(1).
- [26] Oke AS. Convergence of differential transform method for ordinary differential equations. *Journal of Advances in Mathematics and Computer Science*. 2017;24(6):117.

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