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Power Transformations and Unit Mean and Constant Variance Assumptions of the Multiplicative Error Model: The Generalized Gamma Distribution

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Authors' contributions

The first author JO conceived the idea of this paper, however the theoretical results were jointly established by both authors. The second author DCC made the first draft subject to supervision by the first author. Finally both authors read and approved the final manuscript.

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Abstract

Aims: To study the implications of power transformations namely; inverse-square-root, inverse, inverse-square and square transformations on the error component of the multiplicative error and determine whether the unit-mean and constant variance assumptions of the model are either retained or violated after the transformation.
Methodology: We studied the distributions of the error component under the various distributional forms of the generalized gamma distribution namely; Gamma (a, b, 1), Chi-square, Exponential, Weibull, Rayleigh and Maxwell distributions. We first established the functions describing the distributional characteristics of interest for the generalized power transformed error component and secondly applied the unit-mean conditions of the untransformed distributions to the established functions.
Results: We established the following important results in modeling using a multiplicative error model, where data transformation is absolutely necessary;(i) For the inverse-square-root transformation, the unit-mean and constant variance assumptions are approximately maintained for all the distributions under study except the Chi-square distribution where it was violated. (ii) For the inverse transformation, the unit-mean assumptions are violated after the transformation except for the Rayleigh and Maxwell distributions. (iii) For the inverse-square transformation, the unit-mean assumption is violated for all the distributions under study. (iv) For the square transformation, it is only the Maxwell distribution that maintained the unit-mean assumption. (v)

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For all the studied transformations the variances of the transformed distributions were found to be constant but greater than those of the untransformed distribution.
Conclusion: The results of this study though restricted to the distributional forms of the generalized gamma distribution, however they provide a useful framework in modeling for determining where a particular power transformation is successful for a model whose error component has a particular distribution.
Keywords: Error component; mean; multiplicative error model; power transformation; variance.

1 Introduction

A multiplicative error model (MEM) is defined by [1] as

$$X_{t,t\in N} = \mu_t \xi_t \tag{1}$$

where $X_{t,t\in N}$ is a real-valued, discrete time stochastic process defined on [0, + ∞), μ_t , defined conditionally on $\Psi_{t-1} = \mu(\theta, \Psi_{t-1})$ is a positive quantity that evolves deterministically according to the parameter vector, θ. Ψ_{t-1} is the information available for forecasting $X_{t,t\in N}$ and ξ_t is a random variable with a probability density function defined over a [0, +∞) support with unit mean and unknown constant variance, σ_1^2. That is

$$\xi_t \sim V_1^+\left(1, \sigma_1^2\right) \tag{2}$$

There is no question that the distribution of ξ_t in (1) can be specified by means of any probability density function (pdf) having the characteristics in (2). Examples are Gamma, Log-Normal, Weibull, and mixtures of them [1]). [2] favor a Gamma (ϕ,ϕ) (which implies $\sigma^2 = 1/\phi$); [3], in Autoregressive Conditional Duration (ACD) model framework considered a Weilbull $\Gamma\left((1+\phi)^{-1},\phi\right)$ (in this case, $\sigma^2 = \Gamma(1+2\phi)/\Gamma\left((1+\phi)^2-1\right)$). As a result of the above suggested specifications, the error component $\xi_t = \xi$ would be generally studied under the generalized gamma distribution (GGD) which according to [4] can be represented by

$$f(\xi) = \frac{ac(a\xi)^{bc-1} e^{-(a\xi)^c}}{\Gamma(b)}, \ \xi > 0 \tag{3}$$

where a (a shape parameter) and b are real numbers. c can in principle take any real value but normally we consider the case where c ≥ 0. The reason of using the GGD as the study distribution is because the various distributional forms (The 3-parameter gamma, Chi-square, Exponential, Weibull, Rayleigh and Maxwell distributions) of the GGD for various values of a, b and c, have the distributional characteristics given in (2). The distributional forms of (3) for various values of a, b and c are given in Table 1. For more details on the generalized gamma distribution, see [5].

Table 1: Relation of the GGD to other Distributions

S/N	Generalized Gamma Distribution (GG(a, b, c))	a	b	c
1	Gamma $\left(Gamma(a,b,1)\right)$	a	b	1
2	Chi- square	$\frac{1}{2}$	$\frac{n}{2}$	1
3	Exponential	$\frac{1}{\alpha}$	1	1
4	Weibull	$\frac{1}{\sigma}$	1	α
5	Rayleigh	$\frac{1}{\sigma\sqrt{2}}$	1	2
6	Maxwell	$\frac{1}{\sigma\sqrt{2}}$	$\frac{3}{2}$	2

It is not an overstatement to say that statistics is based on various data transformations. Basic statistical summaries such as sample mean, variance, z-scores, histograms, etc., are all transformed data. Some more advanced summaries such as principal components, periodograms, empirical characteristics functions, etc., are also examples of transformed data. According to [6], "transformations in statistics are utilized for several reasons, but unifying arguments are that transformed data"; (i) are easier to report, store and analyze (ii) comply better with a particular modeling framework and (iii) allow for additional insight to the phenomenon not available in the domain of non-transformed data. For example, variance stabilizing transformations, symmetrizing transformations, transformations to additivity, laplace, Fourier, Wavelet, Gabor, Wigner-Ville, Hugh, Mellin, transforms all satisfy one or more of points listed in (i – iii).

Many important results in statistical analysis follow from the assumption that the population being sampled or investigated is normally distributed with a common variance and additive error structure. For the multiplicative error model where normality assumption is out of the question, the assumptions of interest are that the error component has unit mean and constant variance. When the relevant theoretical assumptions relating to a selected method of analysis are approximately satisfied, the usual procedures can be applied in order to make inferences about unknown parameters of interest. In situations where the assumptions are seriously violated several options are available [7]: (i) Ignore the violation of the assumptions and proceed with the analysis as if all assumptions are satisfied. (ii) Decide "what is the correct assumption in place of the one that is violated" and use a valid procedure that takes into account the new assumption. (iii) Design a new model that has important aspects of the original model and satisfies all the assumptions, e.g. by applying a proper transformation to the data or filtering out some suspect data point which may be considered outlying. (iv) Use a distribution-free procedure that is valid even if various assumptions are violated. For more details on the above listed options, see [8].

Most researchers, however, have opted for (iii) which has attracted much attention as documented by [9] and [10] among others. In this study our interest would center on transformation as a remedy for situations where the assumptions for parametric data analysis are seriously violated.

Data transformations are the applications of mathematical modifications to the values of a variable .However caution should be exercised in the choice of the type of transformation to be adopted so that the fundamental structure of the series is not distorted and thereby rendering the interpretation very difficult or impossible. There are two major methods of data transformations namely Bartlett and Box and Cox methods of data transformation, however for ease of application we would only consider the Bartlett's techniques.

[11] used the simple relation between mean and standard deviation over several groups for choice of appropriate transformation. [12] had shown how to apply Bartlett's transformation technique to time series data using the Buys-Ballot table. For details on Buys-Ballot table, see [13]. According to [12], the relationship between variance and mean over several groups is what is needed for choice of appropriate transformation. If we take random samples from a population, the means and standard deviations of these samples will be independent (and thus uncorrelated) if the population has a normal distribution [14]. [12] showed that Bartlett's transformation for time series data is to regress the natural logarithms of the group standard deviations $\left(\hat{\sigma}_{i.}, i=1,2,\ldots,m\right)$ against the natural logarithms of group means $\left(\bar{X}_{i.}, i=1,2,\ldots,m\right)$ and determine the slope, p of the relationship

$$\log_e \hat{\sigma}_{i.} = \alpha + p \log_e \bar{X}_{i.} + error \tag{4}$$

For non-seasonal data that require transformation, we split the observed time series $X_t, t=1,2,\ldots,n$ chronologically into m fairly equal different parts and compute $\left(\bar{X}_{i.}, i=1,2,\ldots,m\right)$ and $\left(\hat{\sigma}_{i.}, i=1,2,\ldots,m\right)$ for the parts. For seasonal data with the length of the periodic intervals, s, the Buys-Ballot table naturally partitions the observed data into m periods or rows for easy application. [12]) also showed that Bartlett's transformation may also be regarded as the power transformation

$$\begin{cases} \log_e X_t, & p=1 \\ X_t^{(1-p)}, & p \neq 1 \end{cases} \tag{5}$$

Summary of transformations for various values of p is given in Table 2.

Table 2: Bartlett's Transformation for some values of p.

S/N	p	Required Transformation
1	0	No transformation
2	$\frac{1}{2}$	$\sqrt{X_t}$
3	1	$\log_e X_t$
4	$\frac{3}{2}$	$\frac{1}{\sqrt{X_t}}$
5	2	$\frac{1}{X_t}$
6	3	$\frac{1}{X_t^2}$
7	-1	X_t^2

Recently there are various studies on the effects of transformation on the error component of the multiplicative error models whose distributional characteristics is given in (2). The overall aim of such studies is to establish the conditions for successful transformation [15]. According to [15], a successful transformation is achieved when the desirable properties of a data set remain unchanged after transformation. For the MEM where the normality assumption of the error component is out of the question, therefore in this study we shall be interested in the unit mean and constant variance assumptions. For the purely multiplicative time series model whose error component in addition to being normally distributed is classified under the characteristics given in (2), [16], [17] and [18] had respectively investigated the effects of logarithm, square and inverse transformations on the error component, ξ_t , where $\xi_t \sim N\left(1,\sigma_1^2\right)$. [16] discovered that the logarithm transform; Y = Log ξ_t can be assumed to be normally distributed with mean, zero and the same variance, σ^2 for $\sigma_1 < 0.1$. Similarly [17] discovered that the square transform; $Y = \xi_t^{\ 2}$ is successful in the interval $0 < \sigma_1 \leq 0.027$, where σ_1 is the standard deviation of the original error component before transformation whereas [18] discovered that the inverse transform $Y = \frac{1}{\xi_t}$ can be assumed to be normally distributed with mean, one and the same variance provided $\sigma \leq 0.10$.

The application of power transformation to model (1) gives

$$X^*_{t,t\in N} = \mu_t^* \xi_t^* \tag{6}$$

where $X^{*}_{t,t\in N} = \left(X_{t,t\in N}\right)^{p} = X^{p}_{t,t\in N}$, $\mu_t^{*} = \mu_t^{\,p}$, $\xi_t^{*} = \left(\xi_t\right)^{p} = \xi_t^{\,p}$ and

$$\xi_t^{\,p} \sim V_2^{+}\left(1, \sigma_2^{2}\right), \; p = -1, \frac{1}{2}, 1, \tfrac{3}{2}, 2, 3 \tag{7}$$

where $\xi_t^{\,p}$ is the generalized power transformed error component of model (1). The most popular power transformations are $\log_e X_t, \sqrt{X_t}, 1/X_t, X_t^{2}$ and $1/X_t^{2}$ .The results of the transformations on model (1) are given in Table 3. The logarithm transformation converts the multiplicative error model (1) to the additive model while the other listed transformations leave the model still multiplicative. For the logarithm transformation

$$Y_t = \log_e X_t = \log_e \mu_t + \log_e \xi_t = \mu_t^{*} + \xi_t^{*} \tag{8}$$

In Table 3 the following notations were adopted

Y_t = Transformed observed series
$\mu_t^{*} =$ The Transformed function of μ_t
$\xi_t^{*} =$ Transformed error component

It is clear from Table 3, that only the logarithm transformation alters the assumptions placed on the error component of the multiplicative error model and as a result interest in this paper would be centered on the transformations that leaves model (1) still multiplicative.

Table 3: Transformations of the multiplicative Error Model

Y_t	μ_t	ξ_t^{*}	Model for Y_t	Assumption on e_t^{*}
$\log_e X_t$	$\log_e \mu_t$	$\log_e \xi_t$	Additive	$\xi_t^{*} \sim V_2^{+}\left(0, \sigma_2^{2}\right)$
$\sqrt{X_t}$	$\sqrt{\mu_t}$	$\sqrt{\xi_t}$	Multiplicative	$\xi_t^{*} \sim V_2^{+}\left(1, \sigma_2^{2}\right)$
$1/X_t$	$1/\mu_t$	$1/\xi_t$	Multiplicative	$\xi_t^{*} \sim V_2^{+}\left(1, \sigma_2^{2}\right)$
X_t^{2}	μ_t^{2}	ξ_t^{2}	Multiplicative	$\xi_t^{*} \sim V_2^{+}\left(1, \sigma_2^{2}\right)$
$1/\sqrt{X_t}$	$1/\sqrt{\mu_t}$	$1/\sqrt{\xi_t}$	Multiplicative	$\xi_t^{*} \sim V_2^{+}\left(1, \sigma_2^{2}\right)$
$1/X_t^{2}$	$1/\mu_t^{2}$	$1/\xi_t^{2}$	Multiplicative	$\xi_t^{*} \sim V_2^{+}\left(1, \sigma_2^{2}\right)$

Since model (6) is still a multiplicative error model and therefore ξ_t^* must also be characterized with unit mean and some constant variance, σ_2^2 which may or may not be equal to σ_1^2. Thus, in this paper, we want to study the effect of power transformations on a non-normal distributed error component of a multiplicative error model whose distributional characteristics belong to the Generalized Gamma distribution. The purpose is to determine if the assumed fundamental structure of the error component (unit mean and constant variance) is maintained after the power transformation and also to investigate what happens to σ_1^2 and σ_2^2 in terms of equality or non-equality. According to [15], the overall reason for concentrating on the error component of model (6) is as plane as the nose on the face: the reason is that the assumptions for model analysis are always placed on the error component.

In this fertile academic minefield, [15] had studied the implication of square root transformation on the unit mean and constant variance assumptions of the error component of model (1) whose distributional characteristics belong to the Generalized Gamma Distribution for the various forms; Chi-square, Exponential, Gamma (a, b, 1), Weibull, Maxwell and Rayleigh distributions. From the results of the study, the unit mean assumption is approximately maintained for all the given distributional forms of the GGD. However there were reductions in the variances of the distributions except those of the Gamma (a, b, 1), for a > 1, Rayleigh and Maxwell that increased, hence they concluded that square-root transformation is not appropriate for multiplicative error model with a Gamma (a, b, 1) for a > 1 or Rayleigh or Maxwell distributed error component. Finally, [15] recommended that square-root transformation, where applicable for a multiplicative error model are successful for the studied distributions if the variance of the transformed error component < 0.5.

Out of the six popular power transformations namely logarithm, square root, inverse, inverse square root, square and inverse square, only the effect of square root transformation on the error component of the multiplicative error model with regard to unit mean and constant variance had been studied by [15], leaving the others yet to be studied. Whereas the logarithm transformation converts the multiplicative error model to an additive model, the others still leave it multiplicative therefore in this paper we study the effect of the power transformations namely; inverse square root, inverse, inverse square and square on the error component of the multiplicative error model with the overall aim of investigating on what happens to the unit mean and constant variance assumptions after the transformations. This study would be carried out under the generalized gamma distribution considering that all the suggested distributions of the error component of the multiplicative error model ([1]; [2];[3]) are the various forms of the generalized gamma distribution.

This paper is organized as follows; Section One contains the introduction while the distributional characteristics of the generalized power transformed error are contained in Section Two. While the results of the study are in Section three, the conclusion, acknowldgements, authors contributions and references are respectively contained in Sections four, five, six and seven..

For simplicity, the notation $\xi_t = \xi$ and $Y_t = Y$ would be adopted.

2 DISTRIBUTIONAL CHARACTERISTICS OF THE GENERALIZED POWER TRANSFORMED ERROR COMPONENT

Suppose the distributional characteristics of ξ belongs to the generalized gamma distribution whose probability density function denoted as $f(\xi)$ is (3), let

$$y = \xi^{p} \tag{9}$$

where p is a power transformation, therefore

$$\xi = y^{\frac{1}{p}}$$

and

$$\frac{d\xi}{dy} = \frac{1}{p} y^{\frac{1}{p}-1}$$

therefore based on the result in [14], the probability density function of y would be given by

$$f(y) = \frac{a^{bc} c\, y^{\frac{1}{p}(bc-1)} e^{-\left(a y^{\frac{1}{p}}\right)^{c}}}{\Gamma(b)} \frac{1}{p} y^{\frac{1}{p}-1}$$

$$= \frac{a^{bc} c\, y^{\left(\frac{bc}{p}-1\right)} e^{-\left(a^{p} y\right)^{\frac{c}{p}}}}{p\Gamma(b)},\ y > 0 \tag{10}$$

(10) is a probability density function (pdf) and in what follows, we have to show that its integral is unity (That is, $\int_0^\infty f(y)\,dy = 1$). We proceed as follows;

$$\int_0^\infty f(y)\,dy = \frac{a^{bc} c}{p\Gamma(b)} \int_0^\infty y^{\left(\frac{bc}{p}-1\right)} e^{-\left(a^{p} y\right)^{\frac{c}{p}}}\,dy \tag{11}$$

By adopting the substitution

$$w = \left(a^{p} y\right)^{\frac{c}{p}},\ 0 < w < \infty \tag{12}$$

in (10) we have the following results

$$y = \frac{w^{\frac{p}{c}}}{a^p}\;;\;\frac{dy}{dw} = \frac{p\,w^{\frac{p}{c}-1}}{c\,a^p};\; dy = \frac{p\,w^{\frac{p}{c}-1}}{c\,a^p}dw \tag{13}$$

Now substituting the results in (12) and (13) into (11) we obtain

$$\int_0^\infty f(y)\,dy = \frac{a^{bc}c}{p\Gamma(b)}\int_0^\infty \left(\frac{w^{\frac{p}{c}}}{a^p}\right)^{\left(\frac{bc}{p}-1\right)} e^{-w}\frac{p\,w^{\frac{p}{c}-1}}{c\,a^p}dw \tag{14}$$

After a series of algebraic evaluation in (14) we have that

$$\int_0^\infty f(y)\,dy = \frac{a^{bc}\left(a^{-p}\right)^{\frac{bc}{p}}\Gamma(b)}{\Gamma(b)} = 1,$$

Having shown that (10) is a proper pdf by showing that its integral is unity, we now proceed to obtain its generalized k[th] moment as follows, where k is a positive integer: By definition $E\left(Y^k\right) = \int_0^\infty y^k\, f(y)\,dy$, hence

$$E\left(Y^k\right) = \int_0^\infty y^k\, f(y)\,dy = \frac{a^{bc}c}{p\Gamma(b)}\int_0^\infty y^k y^{\left(\frac{bc}{p}-1\right)} e^{-\left(a^p y\right)^{\frac{c}{p}}}\,dy$$

$$= \frac{a^{bc}c}{p\Gamma(b)}\int_0^\infty y^{\left(\frac{bc}{p}+k\right)-1} e^{-\left(a^p y\right)^{\frac{c}{p}}}\,dy \tag{15}$$

By applying the substitution in (12) and its results in (13) into (15), we obtain

$$E\left(Y^k\right) = \frac{a^{bc}c}{p\Gamma(b)}\int_0^\infty \left(\frac{w^{\frac{p}{c}}}{a^p}\right)^{\left(\frac{bc}{p}+k-1\right)} e^{-w}\frac{p\,w^{\frac{p}{c}-1}}{c\,a^p}dw$$

$$= \frac{a^{bc}\left(a^{-p}\right)^{\left(\frac{bc}{p}+k\right)}}{\Gamma(b)}\int_0^\infty w^{\left(b+\frac{pk}{c}\right)-1} e^{-w}\,dw$$

$$= \frac{\Gamma\left(b + \frac{pk}{c}\right)}{a^{pk}\Gamma(b)} \quad (16)$$

For k = 1 and 2, we obtain

$$E(Y) = \frac{\Gamma\left(b + \frac{p}{c}\right)}{a^{p}\Gamma(b)} \quad (17)$$

$$E\left(Y^{2}\right) = \frac{\Gamma\left(b + \frac{2p}{c}\right)}{a^{2p}\Gamma(b)} \quad (18)$$

Hence the variance of Y denoted as σ_2^2 is given by

$$\sigma_2^2 = \frac{\Gamma\left(b + \frac{2p}{c}\right)}{a^{2p}\Gamma(b)} - \left[\frac{\Gamma\left(b + \frac{p}{c}\right)}{a^{p}\Gamma(b)}\right]^2 \quad (19)$$

The mean and variance of the untransformed distribution given in (3) had been obtained by [15] as

$$E(\xi) = \frac{1}{a\Gamma(b)}\Gamma\left(b + \frac{1}{c}\right) \quad (20)$$

and

$$\sigma_1^2 = Var(\xi) = E\left(\xi^2\right) - \left(E(\xi)\right)^2 = \frac{1}{a^2\Gamma(b)}\Gamma\left(b + \frac{2}{c}\right) - \left[\frac{1}{a\Gamma(b)}\Gamma\left(b + \frac{1}{c}\right)\right]^2 \quad (21)$$

The means and the variances of the various forms are given in [15]. For the unit mean condition and its impact on the variance for the various forms of the distributions under study as obtained by [15], see Table 4.

Table 4: Condition for Unit Mean and its implication on the Variance of the special cases of the original GGD

Distribution	Mean	Condition for Unit Mean	Variance
Gamma (a, b, 1)	$\frac{b}{a}$	$a = b$	$\frac{1}{a}$
Chi- square	n	$n = 1$	2
Exponential	α	$\alpha = 1$	1.0
Weibull	$\frac{\sigma}{\alpha}\Gamma\left(\frac{1}{\alpha}\right)$	$\alpha = 1 = \sigma$	1.0
Rayleigh	$\sigma\sqrt{\frac{\pi}{2}}$	$\sigma = \sqrt{\frac{2}{\pi}}$	$\frac{1}{\pi}(4-\pi) = 0.3$
Maxwell	$2\sigma\sqrt{\frac{2}{\pi}}$	$\sigma = \frac{1}{2}\sqrt{\frac{\pi}{2}}$	$\frac{3\pi}{8} - 1 = 0.2$

Similarly for the various power transformations under study namely, inverse $\left(p = 2\right)$, inverse square root $\left(p = \frac{3}{2}\right)$, inverse square $\left(p = 3\right)$ and square $\left(p = -1\right)$, the corresponding means and variances of the various distributions under study are obtained and the results are given in Tables 5-8 while the means and variances resulting from the applications of the unit mean conditions of the untransformed distributions to those of the transformed distributions are given in Tables 9-12.

3 RESULTS AND DISCUSSION

Tables 5 through 8 give the theoretical expressions for the mean and variances of the various forms after inverse square root, inverse, inverse square and square transformations respectively. It is important to note that the mean and variance of the exponential distribution as well as the variance of the Rayleigh distribution are undefined (Table 8) for square transformation. In Tables 9 through 12, the means and variances of the transformed distributions resulting from the applications of the unit-mean-conditions for the various forms of the untransformed GGD are respectively given for the various transformations: (Table 9 for inverse-square-root; Table 10 for inverse; Table 11 for inverse-square; Table 12 for square transformations). It is seen in Table 12 that the means and variances resulting from the application of the unit-mean-conditions of the untransformed distribution to the transformed distribution are positively defined for the Gamma (a, b, 1) where a = b > 1, but undefined for the Chi-square, Exponential and Weibull distributions. Furthermore the variance of the Rayleigh distribution is not also defined under this condition.

For the inverse-square-root transformation (Table 9), except the Chi-square distribution that has mean ≈ 2.0 to the nearest whole number, the means of all the other forms under study are approximately unity to the nearest whole number, however, the variances increased. That is

$\sigma_2^2 > \sigma_1^2$ for all the distributions. Here the unit-mean and constant variance assumptions are approximately maintained for all the distributions under study except the Chi-square distribution where the unit mean assumption is violated.

Table 5: Mean and Variance of the special cases of the original GGD under inverse square root transformation $p = \frac{3}{2}$

Distribution	Mean $E(\xi)$	Variance $Var(\xi)$
$Gamma(a,b,1)$	$\frac{\Gamma\left(b+\frac{3}{2}\right)}{a^{\frac{3}{2}}\Gamma(b)}$	$\frac{b(b+1)(b+2)}{a^3} - \left[\frac{\Gamma\left(\frac{2b+3}{2}\right)}{a^{\frac{3}{2}}\Gamma(b)}\right]^2$
Chi- square	$\frac{2^{\frac{3}{2}}\Gamma\left(\frac{n+3}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}$	$n(n+2)(n+4) - 8\left[\frac{\Gamma\left(\frac{n+3}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\right]^2$
Exponential	$\frac{3}{4}\alpha^{\frac{3}{2}}\Gamma\left(\frac{1}{2}\right)$	$3\alpha^3\left[8 - \frac{3}{16}\left[\Gamma\left(\frac{1}{2}\right)\right]^2\right]$
Weibull	$\sigma^{\frac{3}{2}}\frac{3}{2\alpha}\Gamma\left(\frac{3}{2\alpha}\right)$	$\sigma^3\frac{3}{\alpha}\left[\Gamma\left(\frac{3}{\alpha}\right) - \frac{3}{4\alpha}\left(\Gamma\left(\frac{3}{2\alpha}\right)\right)^2\right]$
Rayleigh	$\sigma^{\frac{3}{2}}2^{\frac{3}{4}}\Gamma\left(\frac{7}{4}\right)$	$\sigma^3 2^{\frac{3}{2}}\left[\Gamma\left(\frac{5}{2}\right) - \left(\Gamma\left(\frac{7}{4}\right)\right)^2\right]$
Maxwell	$\sigma^{\frac{3}{2}}2^{\frac{3}{4}}\frac{\Gamma\left(\frac{9}{4}\right)}{\Gamma\left(\frac{3}{2}\right)}$	$\sigma^3 2^{\frac{3}{2}}\left[\frac{4}{\Gamma\left(\frac{1}{2}\right)} - \left(\frac{\Gamma\left(\frac{9}{4}\right)}{\Gamma\left(\frac{3}{2}\right)}\right)^2\right]$

For the inverse transformation (Table 10), only the means of the Rayleigh and Maxwell distributions are approximately 1.0, others have mean ≥ 2.0 to the nearest whole number. Also $\sigma_2^2 > \sigma_1^2$ for all the distributions. Here the unit-mean assumptions are violated except for the Rayleigh and Maxwell distributions.

Furthermore, for the inverse-square transformation (Table 11), they were increases in the variances, $\sigma_2^2 > \sigma_1^2$ for all the distributions however the means are all ≥ 2.0 to the nearest whole number. Under this transformation, the unit-mean assumption is violated for all the distributions under study.

Finally, for the square transformation (Table 12), either the mean or the variance or both are undefined for the various distributions except the Maxwell distribution that maintained the unit mean even though its variance increased after the transformation. Here it is only the Maxwell distribution that maintained the unit-mean assumption.

Table 6: Mean and Variance of the special cases of the original GGD under inverse transformation $p=2$

Distribution	Mean $E(\xi)$	Variance $Var(\xi)$
$Gamma(a,b,1)$	$\frac{b(b+1)}{a^2}$	$\frac{b(b+1)}{a^4}\left[(b+2)(b+3)-b(b+1)\right]$
Chi- square	$n(n+2)$	$n(n+2)\left[(n+4)(n+6)-n(n+2)\right]$
Exponential	$2\alpha^3$	$4\alpha^4(6-\alpha^2)$
Weibull	$\frac{2\sigma^2}{\alpha}\Gamma\left(\frac{2}{\alpha}\right)$	$\frac{4\sigma^4}{\alpha}\left[\Gamma\left(\frac{4}{\alpha}\right)-\frac{1}{\alpha}\left(\Gamma\left(\frac{2}{\alpha}\right)\right)^2\right]$
Rayleigh	$2\sigma^2$	$4\sigma^4$
Maxwell	$3\sigma^2$	$6\sigma^4$

Table 7: Mean and Variance of the special cases of the original GGD under inverse square transformation $p=3$

Distribution	Mean $E(\xi)$	Variance $Var(\xi)$
$Gamma(a,b,1)$	$\frac{b(b+1)(b+2)}{a^3}$	$\frac{b(b+1)(b+2)}{a^6}\left[(b+3)(b+4)(b+5)-b(b+1)(b+2)\right]$
Chi- square	$n(n+2)(n+4)$	$n(n+2)(n+4)\left[(n+6)(n+8)(n+10)-n(n+2)(n+4)\right]$
Exponential	$6\alpha^3$	$684\alpha^6$
Weibull	$\frac{3\sigma^3}{\alpha}\Gamma\left(\frac{3}{\alpha}\right)$	$\frac{3\sigma^6}{\alpha}\left[2\Gamma\left(\frac{6}{\varepsilon}\right)-\frac{3}{\alpha}\left(\Gamma\left(\frac{3}{\alpha}\right)\right)^2\right]$
Rayleigh	$\frac{3\sigma^3}{\sqrt{2}}\Gamma\left(\frac{1}{2}\right)$	$3\sigma^6\left[16-\frac{3}{2}\left(\Gamma\left(\frac{1}{2}\right)\right)^2\right]$
Maxwell	$\frac{8\sigma^3\sqrt{2}}{\Gamma\left(\frac{1}{2}\right)}$	$\sigma^6\left[105-\frac{128}{\left[\Gamma\left(\frac{1}{2}\right)\right]^2}\right]$

Table 8: Mean and Variance of the special cases of the original GGD under square transformation $p=-1$

Distribution	Mean $E(\xi)$	Variance $Var(\xi)$
$Gamma(a,b,1)$	$\frac{a}{b-1}$	$\frac{a^2}{b-1}\left[\frac{1}{(b-1)}-\frac{1}{(b-2)}\right]$
Chi- square	$\frac{1}{n-2}$	$\frac{1}{n-2}\left[\frac{1}{(n-4)}-\frac{1}{(n-2)}\right]$
Exponential	Undefined	Undefined
Weibull	$\frac{1}{\sigma}\Gamma\left(1-\frac{1}{\alpha}\right),\alpha>1$	$\frac{1}{\sigma^2}\left[\Gamma\left(1-\frac{2}{\alpha}\right)-\left(\Gamma\left(1-\frac{1}{\alpha}\right)\right)^2\right],\alpha>2$
Rayleigh	$\frac{1}{\sigma\sqrt{2}}\Gamma\left(\frac{1}{2}\right)$	Undefined
Maxwell	$\frac{\sqrt{2}}{\sigma\Gamma\left(\frac{1}{2}\right)}$	$\frac{1}{\sigma^2}\left(1-\frac{2}{\left(\Gamma\left(\frac{1}{2}\right)\right)^2}\right)$

4 CONCLUSION

In this study, we investigated the implication of power transformations namely, inverse-square-root, inverse, inverse-square and square transformations on the unit-mean and constant variance assumptions of the error component of the multiplicative error model. The distributions of the error component studied were the various forms of the generalized gamma distribution namely Gamma (a, b, 1), Chi-square, Exponential, Weibull, Rayleigh and Maxwell distributions. The purpose of the study is to investigate on whether the unit-mean and constant variance assumptions necessary for modeling using the multiplicative error model are either violated or retained after the various power transformations. Firstly, the functions describing distributional characteristics of interest for the generalized power transformed error component were established and secondly the unit-mean conditions of the untransformed distributions were applied to the established functions with a view to studying their impacts on the transformed distribution. From the results of the study, the following were discovered;

(i) For the inverse-square-root transformation (Table 9), except the Chi-square distribution that has mean ≈ 2.0 to the nearest whole number and the Gamma(a,b,1) whose mean and variance depends on the parameter a (a=b), the means of all the other forms under study are approximately unity. However, the variances increased. That is $\sigma_2^2 > \sigma_1^2$ for all the distributions. Here the unit-mean and constant variance assumptions are approximately maintained for all the distributions under study except the Chi-square distribution where it is violated.

(ii) For the inverse transformation (Table 10), only the means of the Rayleigh and Maxwell distributions are approximately 1.0, others have mean ≥ 2.0 to the nearest whole number. Also $\sigma_2^2 > \sigma_1^2$ for all the distributions. Here the unit-mean assumptions are violated except for the Rayleigh and Maxwell distributions.

(iii) For the inverse-square transformation (Table 11), there were increases in the variances after transformation. That is $\sigma_2^2 > \sigma_1^2$ for all the distributions however the means are all ≥ 2.0 to the nearest whole number. Under this transformation, the unit-mean assumption is violated for all the distributions under study.

(iv) For the square transformation (Table 12), either the mean or the variance or both are undefined under the application of the unit mean condition for the Chi-square, Exponential, Weibull and Rayleigh distributions however the Maxwell distribution maintained the unit mean assumption even though its variance increased after the transformation. Here it is only the Maxwell distribution that maintained the unit-mean assumption. Finally it is important to note that under this transformation the mean and variance of the Gamma (a,b,1) are positively defined for a >1.

Table 9: Application of the Unit Mean condition of the original GGD and its implications on the Mean and Variance of the special cases under inverse square root transformation $p = \frac{3}{2}$

Distribution	Unit mean of the Untransformed Distribution	Mean $E(\xi)$	Variance $Var(\xi)$
$Gamma(a,b,1)$	b = a	$\frac{\Gamma\left(a+\frac{3}{2}\right)}{a^{\frac{3}{2}}\Gamma(a)}$	$\frac{(a+1)(a+2)}{a^2} - \left[\frac{\Gamma\left(a+\frac{3}{2}\right)}{a^{\frac{3}{2}}\Gamma(a)}\right]^2$
Chi- square	n = 1	$\frac{2^{\frac{1}{2}}\Gamma(2)}{\Gamma\left(\frac{1}{2}\right)} = 1.6$	$n(n+2)(n+4) - 8\left[\frac{\Gamma\left(\frac{n+3}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\right]^2 = 12.5$
Exponential	$\alpha = 1$	$\frac{3}{4}\Gamma\left(\frac{1}{2}\right) = 1.3$	$3\alpha^3\left[8 - \frac{3}{16}\left[\Gamma\left(\frac{1}{2}\right)\right]^2\right] = 22.2$
Weibull	$\alpha = \sigma = 1$	$\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = 1.3$	$\sigma^3\frac{3}{\alpha}\left[\Gamma\left(\frac{3}{\alpha}\right) - \frac{3}{4\alpha}\left(\Gamma\left(\frac{3}{2\alpha}\right)\right)^2\right] = 4.2$
Rayleigh	$\sigma = \sqrt{\frac{2}{\pi}}$	$\left(\sqrt{\frac{2}{\pi}}\right)^{\frac{3}{2}} 2^{\frac{3}{4}}\Gamma\left(\frac{7}{4}\right) = 1.2$	$\sigma^3 2^{\frac{3}{2}}\left[\Gamma\left(\frac{5}{2}\right) - \left(\Gamma\left(\frac{7}{4}\right)\right)^2\right] = 0.7$
Maxwell	$\sigma = \frac{1}{2}\sqrt{\frac{2}{\pi}}$	$\left(\frac{1}{2}\sqrt{\frac{2}{\pi}}\right)^{\frac{3}{2}} 2^{\frac{3}{4}}\frac{\Gamma\left(\frac{9}{4}\right)}{\Gamma\left(\frac{3}{2}\right)} = 0.5$	$\sigma^3 2^{\frac{3}{2}}\left[\frac{4}{\Gamma\left(\frac{1}{2}\right)} - \left(\frac{\Gamma\left(\frac{9}{4}\right)}{\Gamma\left(\frac{3}{2}\right)}\right)^2\right] = 0.2$

Table 10: Application of the Unit Mean condition of the original GGD and its implications on the Mean and Variance of the special cases under inverse transformation $p=2$

S/n	Distribution	Mean $E(\xi)$	Variance $Var(\xi)$
1	$Gamma(a,b,1)$	$\frac{(a+1)}{a}$	$\frac{(a+1)}{a^3}\left[(a+2)(a+3)-a(a+1)\right]$
2	Chi- square	3	$n(n+2)\left[(n+4)(n+6)-n(n+2)\right]$ $=96.0$
3	Exponential	$2\alpha^3=2.0$	$4\alpha^4\left(6-\alpha^2\right)=1.0$
4	Weibull	$\frac{2\sigma^2}{\alpha}\Gamma\left(\frac{2}{\alpha}\right)$ $=2.0$	$\frac{4\sigma^4}{\alpha}\left[\Gamma\left(\frac{4}{\alpha}\right)-\frac{1}{\alpha}\left(\Gamma\left(\frac{2}{\alpha}\right)\right)^2\right]$ $=20.0$
5	Rayleigh	$2\sigma^2=1.3$	$4\sigma^4=1.6$
6	Maxwell	$3\sigma^2=1.2$	$6\sigma^4=0.9$

Table 11: Application of the Unit Mean condition of the original GGD and its implications on the Mean and Variance of the special cases under inverse square transformation $p=3$

Distribution	Mean $E(\xi)$	Variance $Var(\xi)$
$Gamma(a,b,1)$	$\frac{(a+1)(a+2)}{a^2}$	$\frac{(a+1)(a+2)}{a^5}\left[(a+3)(a+4)(a+5)-a(a+1)(a+2)\right]$
Chi- square	$n(n+2)(n+4)$ $=15.0$	$n(n+2)(n+4)\left[(n+6)(n+8)(n+10)-n(n+2)(n+4)\right]$ $=10170$
Exponential	$6\alpha^3=6.0$	$\frac{3\sigma^6}{\alpha}\left[2\Gamma\left(\frac{6}{\alpha}\right)-\frac{3}{\alpha}\left(\Gamma\left(\frac{3}{\alpha}\right)\right)^2\right]=684.0$
Weibull	$\frac{3\sigma^3}{\alpha}\Gamma\left(\frac{3}{\alpha}\right)=6.0$	$684\alpha^6=684.0$
Rayleigh	$\frac{3\sigma^3}{\sqrt{2}}\Gamma\left(\frac{1}{2}\right)=3.8$	$3\sigma^6\left[16-\frac{3}{2}\left(\Gamma\left(\frac{1}{2}\right)\right)^2\right]=0.9$
Maxwell	$\frac{8\sigma^3\sqrt{2}}{\Gamma\left(\frac{1}{2}\right)}=1.6$	$\sigma^6\left(105-\frac{128}{\left(\Gamma\left(\frac{1}{2}\right)\right)^2}\right)=3.9$

Table 12: Application of the Unit Mean condition of the original GGD and its implications on the Mean and Variance of the special cases under square transformation $p=-1$

Distribution	Mean $E(\xi)$	Variance $Var(\xi)$
$Gamma(a,b,1)$	$\frac{a}{a-1}$	$\frac{a^2}{a-1}\left[\frac{1}{(a-1)}-\frac{1}{(a-2)}\right]$
Chi- square	*	*
Exponential	*	*
Weibull	*	*
Rayleigh	$\frac{1}{\sigma\sqrt{2}}\Gamma\left(\frac{1}{2}\right)=1.0$	*
Maxwell	$\frac{\sqrt{2}}{\sigma\Gamma\left(\frac{1}{2}\right)}=1.3$	$\frac{1}{\sigma^2}\left[1-\frac{2}{\left(\Gamma\left(\frac{1}{2}\right)\right)^2}\right]=0.9$

Note: * means undefined under the application of the unit mean condition of the untransformed distribution

Finally, except for the Gamma (a,b,1) which under the application of the unit mean condition has mean and variance that depend on a where a =b, we make the following recommendations based on the results of this study;

(i) Inverse-square-root transformation is appropriate for Exponential, Weibull, Rayleigh and Maxwell distributed error components.
(ii) Inverse transformation is appropriate for a data set whose error component belongs to Rayleigh or Maxwell distributions.
(iii) Inverse square transformation is not appropriate for a data set whose error component belong to the six studied distributions.
(iv) Square transformation is only appropriate for a data set whose error component belongs to a Maxwell distribution.

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Competing Interests

Authors have declared that no competing interests exist.

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